Review of Linear Accelerator Optics

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- Outline:
  - **Transverse optics**
  - **Longitudinal optics**
  - **Radiation**

Concepts

Want to touch on a number of concepts including:

- **Closed orbit**
- **Betatron tune**
- **Dispersion**
- **Momentum compaction**
- **Transfer matrix**
- **Twiss parameters and phase advance**
- **Chromaticity**
- **Synchrotron Radiation**
- **Energy spread**
- **(Equilibrium) Emittance**
- **Synchrotron Oscillations**

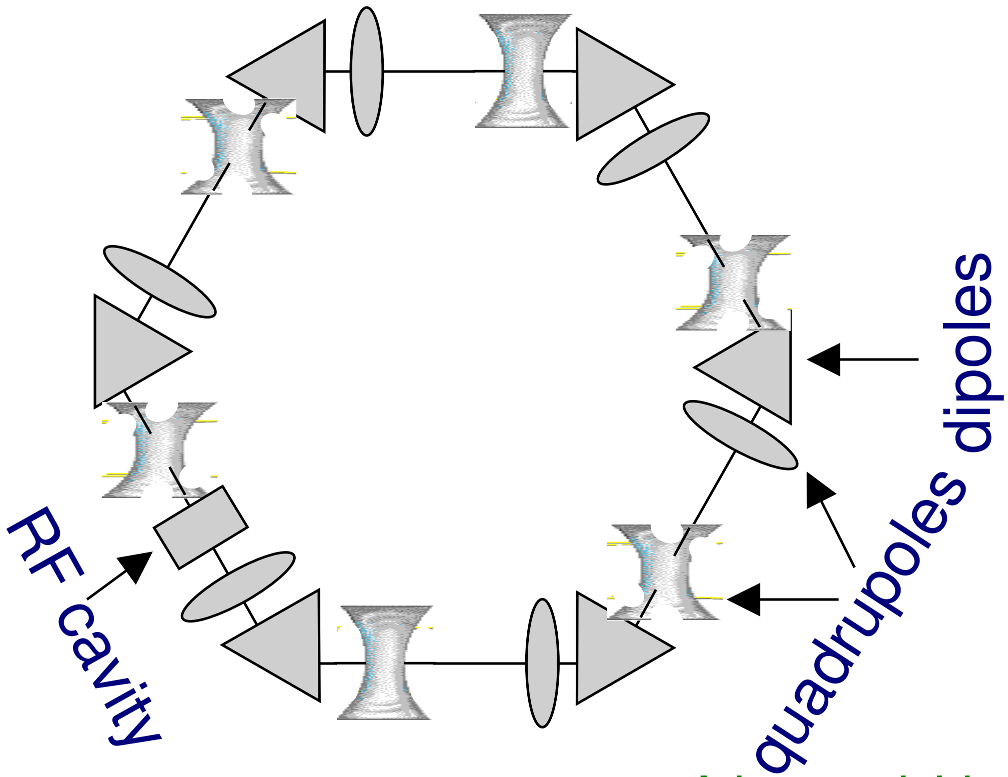
Transverse Beamdynamics Terminology

- Linear beamdynamics determined by: Dipoles, Quadrupoles (lenses), Solenoids, rf-resonators, (synchrotron radiation)
- Nonlinear: sextupoles, higher multipoles, errors, insertion devices (undulators/wigglers), stochastic nature of SR, …
- Trajectory/Orbit – (single pass/periodic)
- Ideal orbit: through the centers of all (ideally aligned) elements
- Closed orbit: closed, periodic trajectory around a ring (closes after one turn in position and angle).
- Particles that deviate from the closed orbit will oscillate about it (transverse: Betatron oscillations)
- Can be described by so-called Hill equation. So-called pseudo-harmonic oscillator.

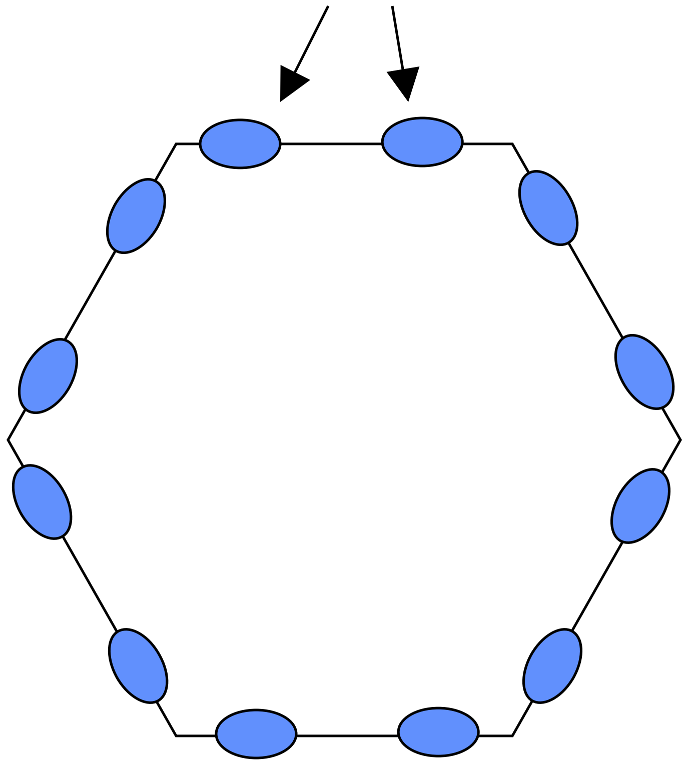
Particle Storage Rings

In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.

Optics elements



Particle bunches



Equations of Motion in a Storage Ring

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m **is the relativistic mass of the particle,**

e **is the charge of the particle,**

v **is the velocity of the particle,**

a **is the acceleration of the particle,**

E **is the electric field and,**

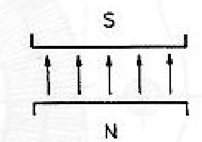
B **is the magnetic field.**

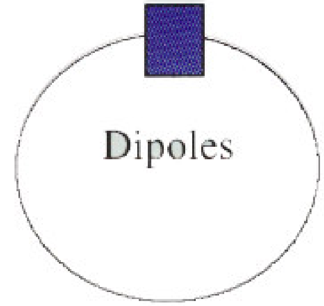
Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

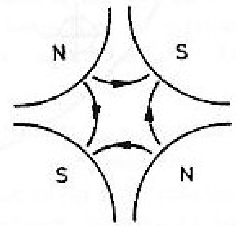
$$B_x = 0$$
$$B_y = B_o$$

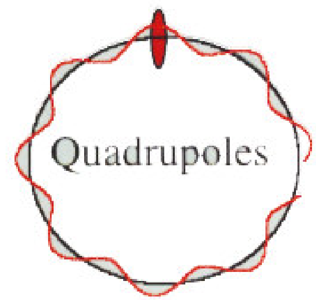




Quadrupoles → used for focussing

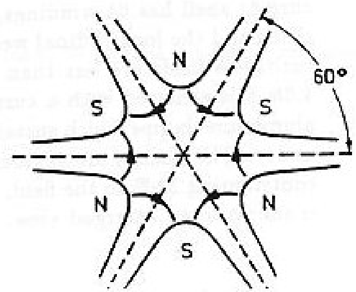
$$B_x = Ky$$
$$B_y = -Kx$$

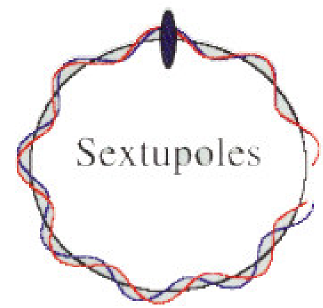




Sextupoles → used for chromatic correction

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$

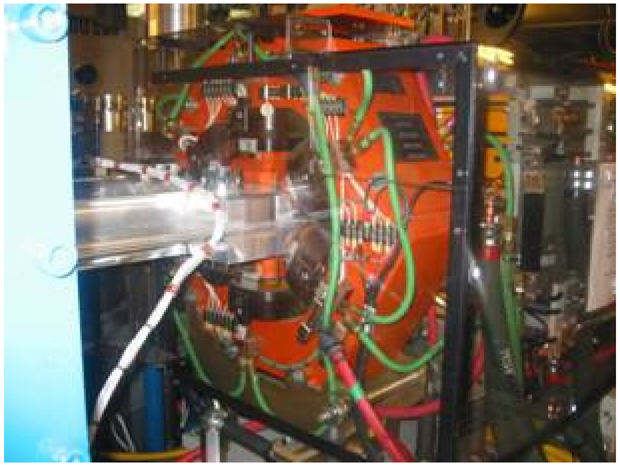




Practical Magnet Examples at the ALS


Quadrupoles

Dipoles


Sextupoles

Two approaches

There are two approaches to introduce the motion of particles in a storage ring

1. The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, …

2. The way that our computer models actually do it

I will begin with the second way and then go back to the first.

Equations of motion

Begin with equations of motion → Lorentz force

↓

Change dependent variable from time to longitudinal position

↓

Integrate particle trajectory around the ring and find the closed orbit

↓

Generate a map around the closed orbit

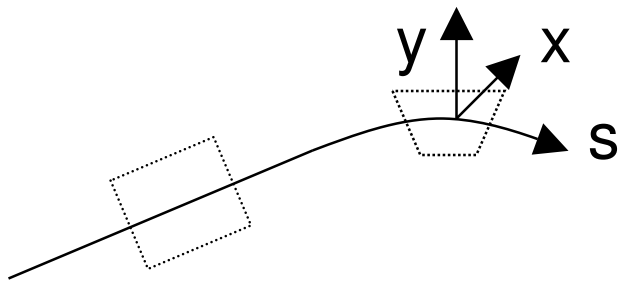
↓

Analyze and track the map around the ring

Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



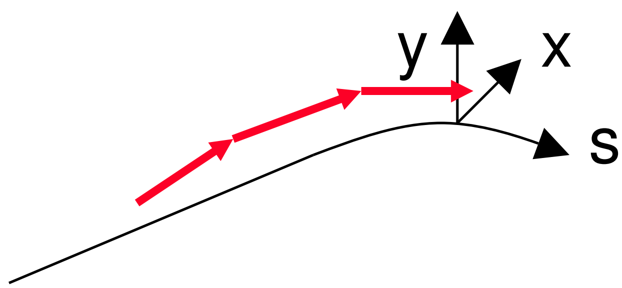
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

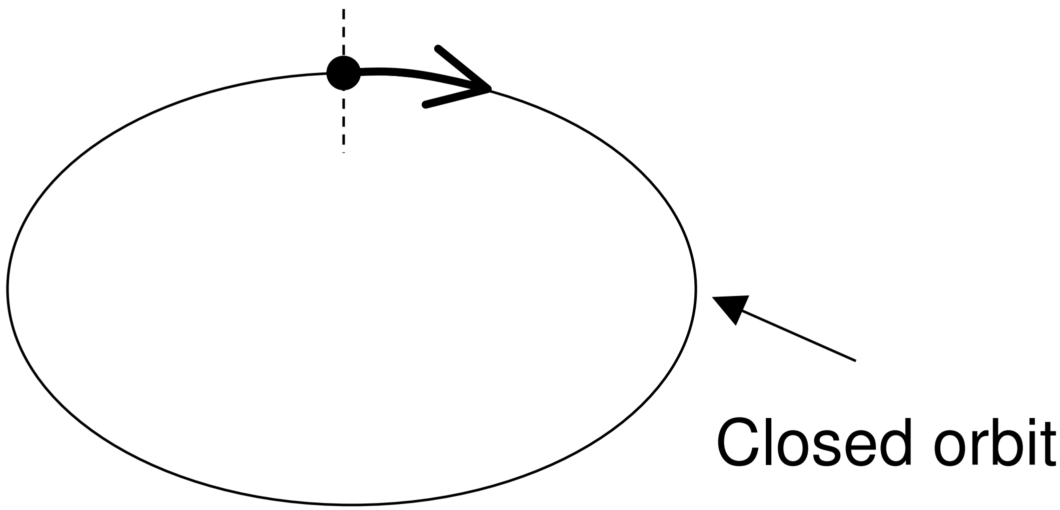
$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$



***Note sometimes one uses canonical momentum rather than x' and y'**

Find the Closed Orbit

A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.

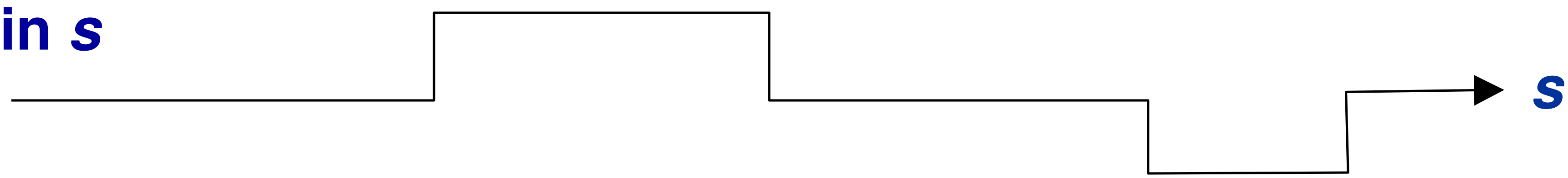


In every working story ring there exists at least one closed orbit.

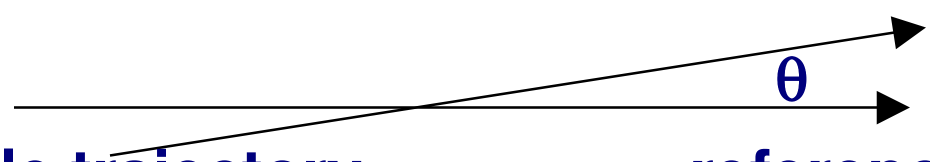
Approximation

Everything up to now there was general. No discussion of the field representation or the integrator. In many codes simplifications are made.

1. The velocity of the particle is the speed of light → $v = c$
2. The magnetic field is isomagnetic. Piecewise constant in s



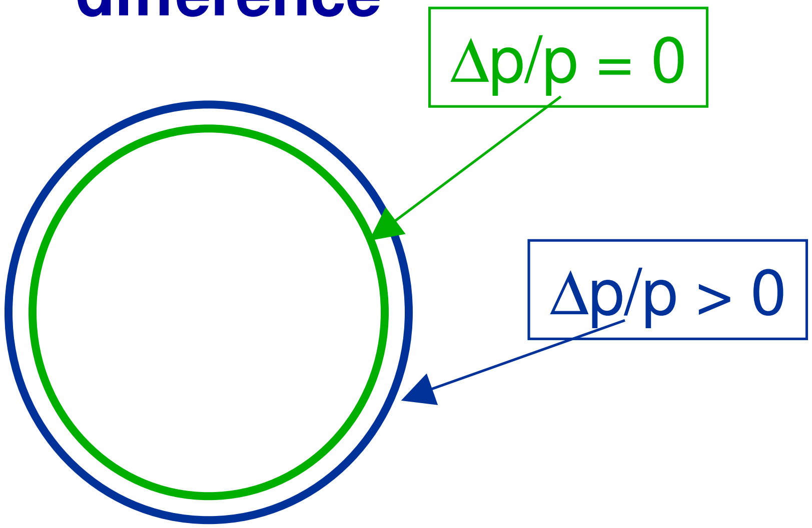
3. The angle of the particles with respect to the reference particle is small and can assume that $\theta = \tan\theta$



Dispersion and momentum compaction

Assume that the energy is fixed → no cavity or damping

- **Find the closed orbit for a particle with slightly different energy than the nominal particle. The dispersion is the difference in closed orbit between them normalized by the relative momentum difference**



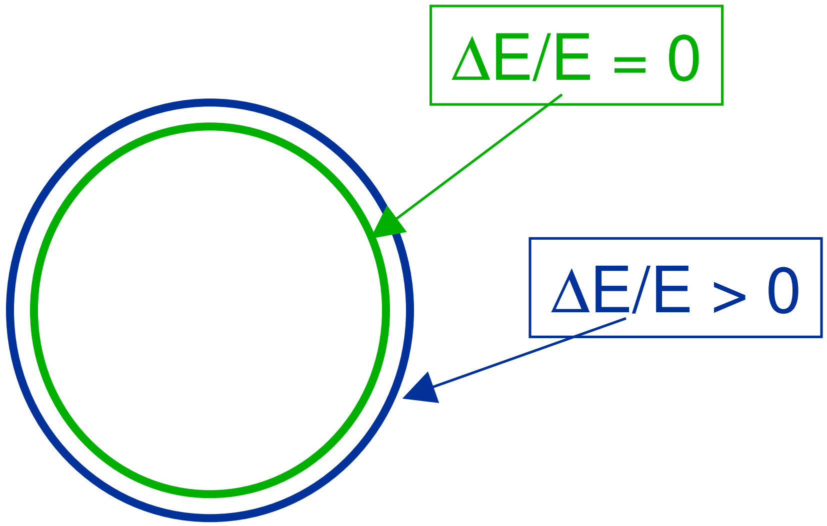
$$x = D_x \frac{\Delta p}{p}, y = D_y \frac{\Delta p}{p}$$

$$x' = D'_x \frac{\Delta p}{p}, y' = D'_y \frac{\Delta p}{p}$$

Momentum Compaction

Momentum compaction, α, is the change in the closed orbit length as a function of momentum.

$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$



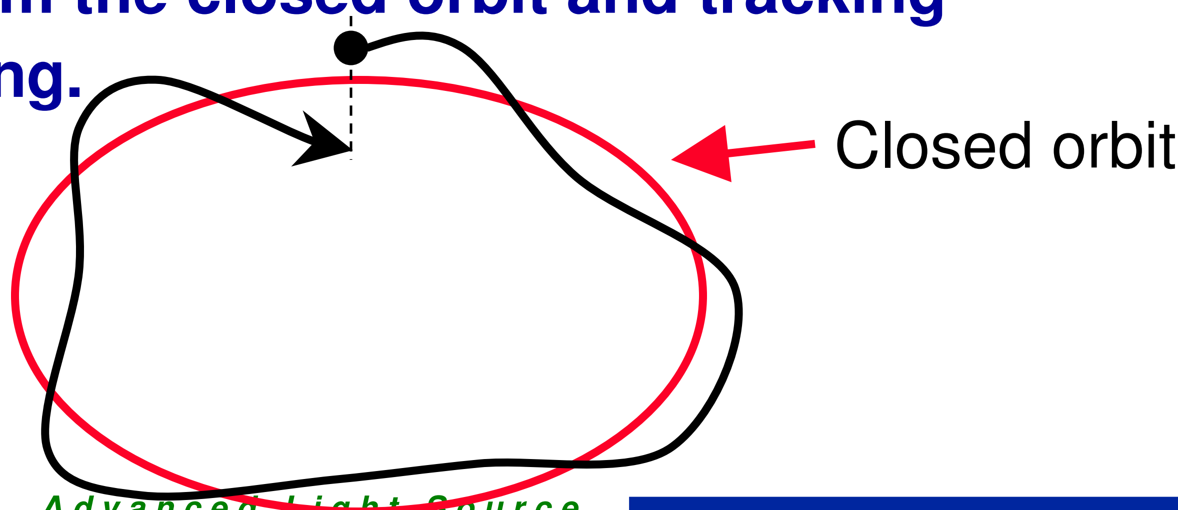
Generate a one-turn Map Around the Closed Orbit

A one-turn map, *R*, maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

$$x_f = x_i + \frac{dx_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

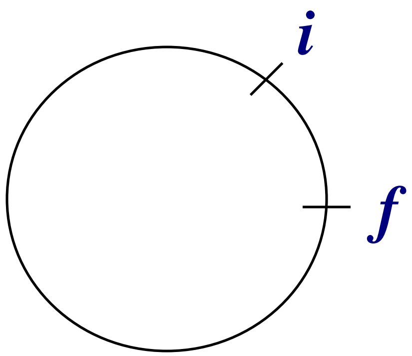
$$x'_f = x'_i + \frac{dx'_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx'_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



Transfer Matrix

One can write the linear transformation between one point in the storage ring (i) to another point (f) as



$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

this is for the case of uncoupled horizontal motion. One can extend this to 4x4 or 6x6 cases.

Examples of transfer matrices

Drift of length L

$$R_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

The matrix for a focusing quadrupole of gradient $k = (\partial B / \partial x)/(B\rho)$ and of length l_q

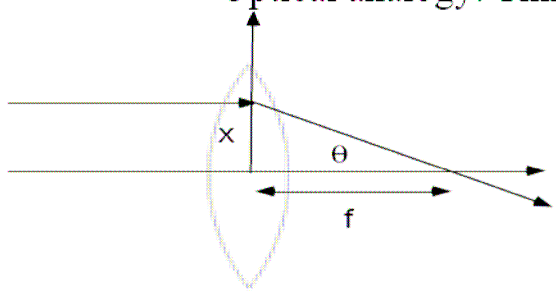
$$R_{Quad} = \begin{pmatrix} \cos\phi & \sin\phi / \sqrt{|k|} \\ -\sqrt{|k|}\sin\phi & \cos\phi \end{pmatrix}$$

The matrix for a zero length thin quadrupole $K = |k|\, l_q$

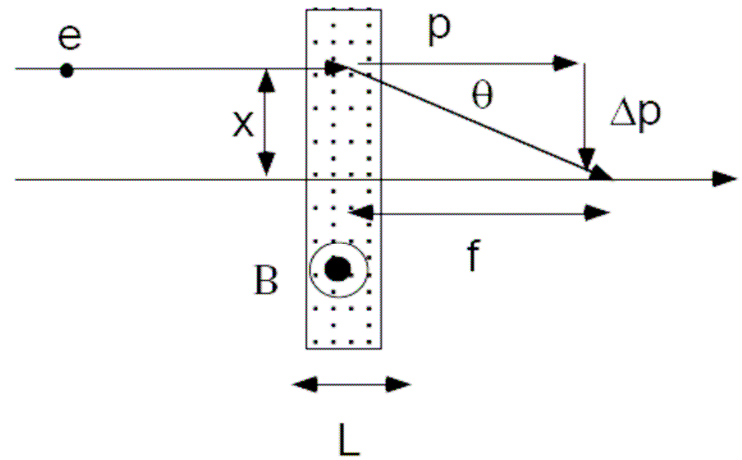
$$R_{thin-lens} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}$$

Magnetic lenses: Quadrupoles

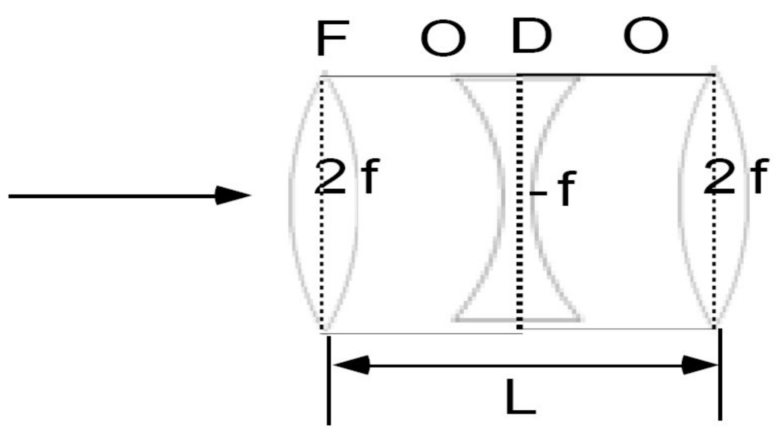
Magnetic focusing fields:
Optical analogy: Thin lens, focal length f



$$\theta = \frac{x}{f}$$



- Thin lens representation/FODO cell



$$\begin{pmatrix} x(s,s_0) \\ x'(s,s_0) \\ y(s,s_0) \\ y'(s,s_0) \\ \delta_l(s,s_0) \\ \delta \end{pmatrix}$$

Thin lens:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix}$$

Drift:

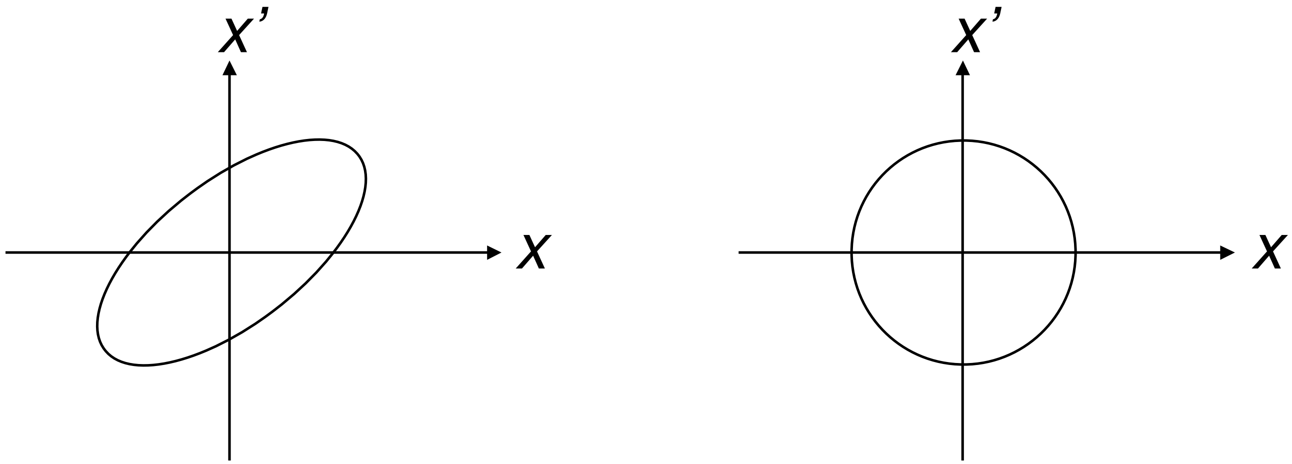
$$\begin{pmatrix} 1 & L/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Computation of beta-functions and tunes

One can diagonalize the one-turn matrix, ***R***

$$N_{one-turn} = A R_{one-turn} A^{-1}$$

This separates all the global properties of the matrix into ***N*** and the local properties into ***A***.



In the case of an uncoupled matrix the position of the particle each turn in x-x' phase space will lie on an ellipse. At different points in the ring the ellipse will have the same area but a different orientation.

Computation of beta-functions and tunes

The eigen-frequencies are the tunes. *A* contains information about the beam envelope. In the case of an uncoupled matrix one can write *A* and *R* in the following way:

$$N_{one-turn} = AR_{one-turn}A^{-1}$$

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\phi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

The beta-functions can be propagated from one position in the ring to another by tracking *A* using the transfer map between the initial point the final point

$$A_f = R_{fi}A_i$$

This is basically how our computer models do it.

First approach – traditional one

This approach provides some insights but is limited

Begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2x}{ds^2} = -k(s)x,$$

where $k = \dfrac{B_T}{(B\rho)a}$**, with**

B_T **being the pole tip field**

a **the pole-tip radius, and**

$B\rho$**[T-m]** $\approx 3.356\,p$**[GeV/c]**

Hills equation

The solution can be parameterized by a psuedo-harmonic oscillation of the form

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

where $\beta(s)$ is the beta function,

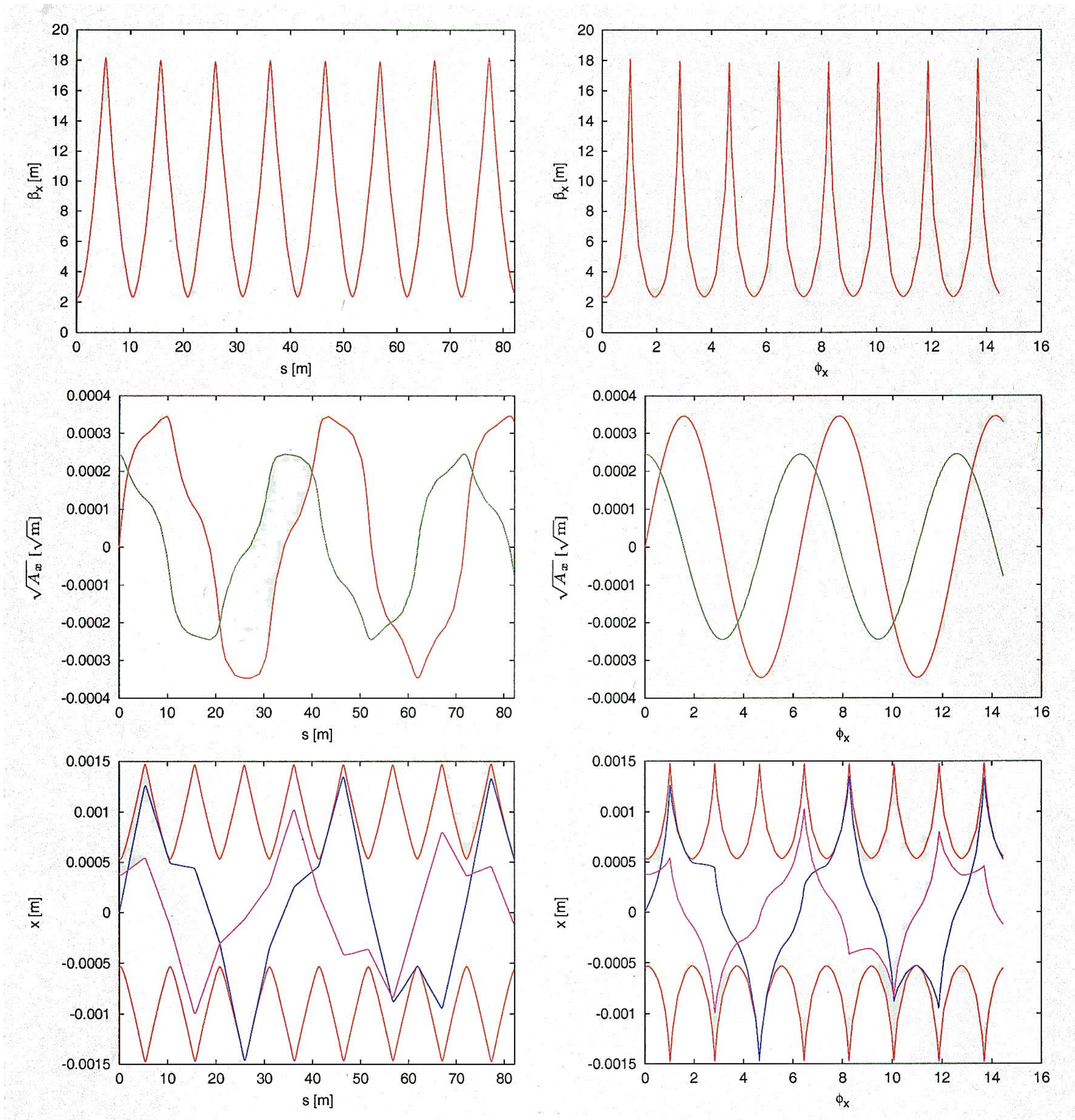
$\alpha(s)$ is the alpha function,

$\varphi_{x,y}(s)$ is the betatron phase, and

ε is an action variable

$$\varphi = \int_0^s \frac{ds}{\beta}$$

Example of Twiss parameters and trajectories



Twiss Parameters and Phase Advance

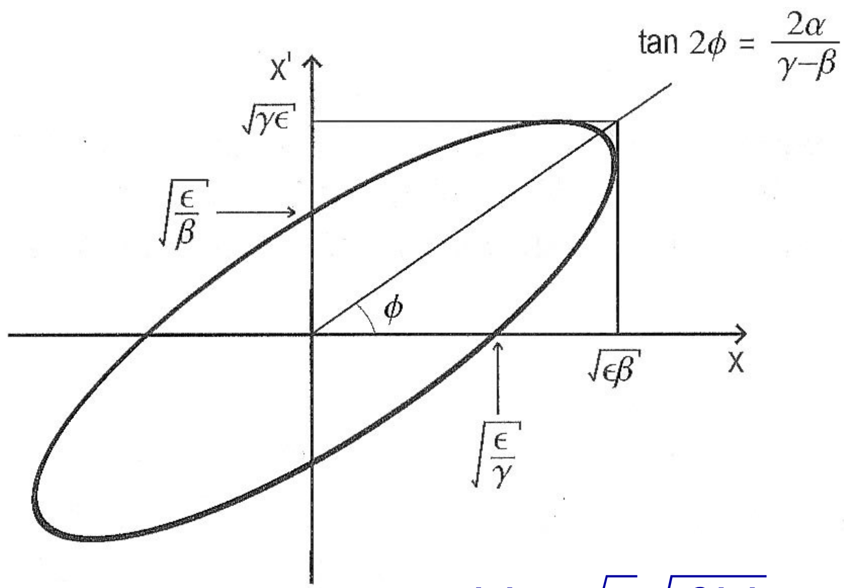
In addition to β there is α and γ:

$$\alpha = -\frac{\beta'}{2},$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ε:

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s) + \varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s) + \varphi_0)$$

Emittance Definition/Statistical

- Emittance defined as the phase space area occupied by an ensemble of particles
- Phase space means consisting of pairs of position and (canonical) momentum variables
- Example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).
- Emittance can be defined as a statistical quantity (beam is composed of finite number of particles)

$$\varepsilon_{geometric,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

- In certain systems (I will not go into the mathematical details) the (normalized) emittance is a conserved quantity – e.g. single charged particle traveling down a magnetic structure – Liouville.

Transport of the beam ellipse

Beam ellipse matrix

$$\sum_{beam}^{x} = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum_{beam,f}^{x} = R_{x,i-f} \sum_{beam,i}^{x} R_{x,i-f}^{T}$$

Transport of the beam ellipse

Transport of the twiss parameters in terms of the transfer matrix elements

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_f = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & 1+C'S & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix}\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_i$$

Transfer matrix can be expressed in terms of the twiss parameters and phase advances

$$R_{fi} = \begin{pmatrix} \sqrt{\frac{\beta_f}{\beta_i}}\left(\cos\varphi_{fi} + \alpha_i \sin\varphi_{fi}\right) & \sqrt{\beta_f \beta_i}\sin\varphi_{fi} \\ -\frac{1+\alpha_i\alpha_f}{\sqrt{\beta_f\beta_i}}\sin\varphi_{fi} + \frac{\alpha_i - \alpha_f}{\sqrt{\beta_f\beta_i}}\cos\varphi_{fi} & \sqrt{\frac{\beta_i}{\beta_f}}\left(\cos\varphi_{fi} - \alpha_f \sin\varphi_{fi}\right) \end{pmatrix}$$

One turn matrix

The one turn matrix can be written

$$R_{one-turn} = \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

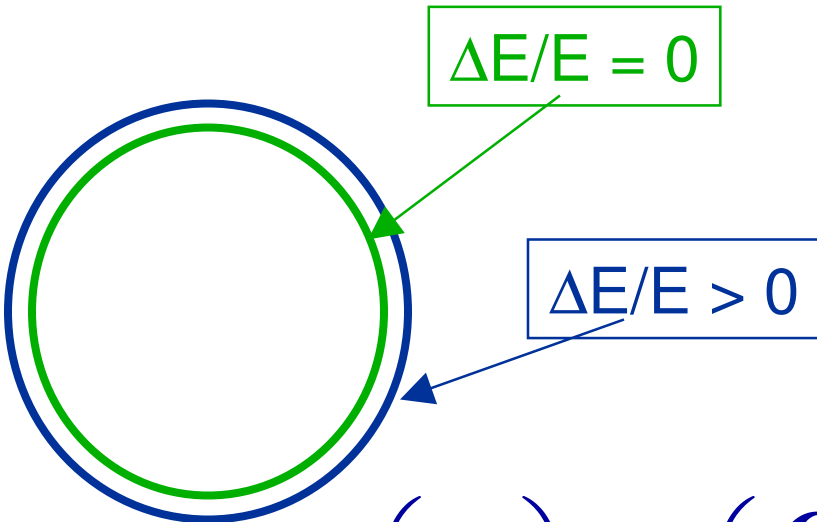
Where the betatron tune, $\nu = \phi/(2*\pi)$

By diagonalizing the one turn matrix one can separate the global quantities (such as tune) from the local quantities such as β.

$$\begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\phi & \cos\varphi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

Dispersion

Dispersion, *D*, is the change in closed orbit as a function of energy

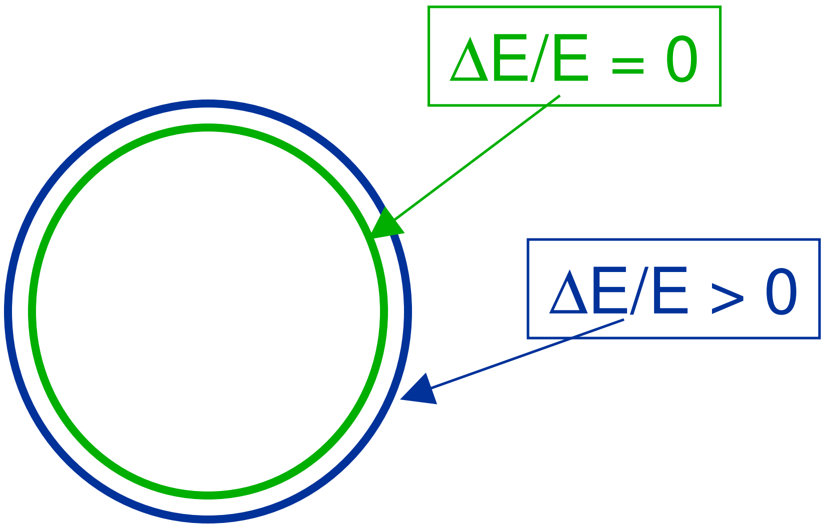


$$x = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_i$$

Momentum Compaction

Momentum compaction, α, is the change in the closed orbit length as a function of energy.



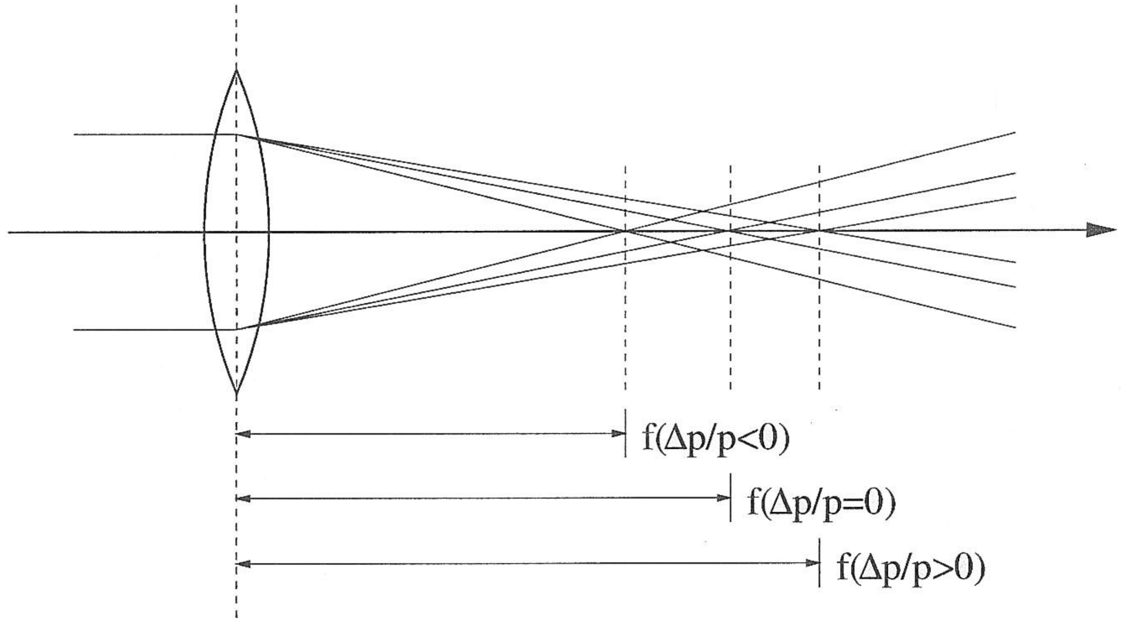
$$\frac{\Delta L}{L} = \alpha \frac{\Delta E}{E}$$

$$\alpha = \int_0^{L_0} \frac{D_x}{\rho} ds$$

Isochronicity/Transition Energy

- If $\dfrac{1}{\gamma^2} = \alpha$,
- the circulation time does not depend on the particle momentum any more. One calls this situation isochronous transport
- Has relevance both for transfer lines
  - Potentially maximizes energy gain
  - Minimizes time dispersion, i.e. short bunches stay short
- And storage rings
  - Minimum bunch length
  - Synchrotron oscillations (see later) ‘freeze’
    - Transition energy

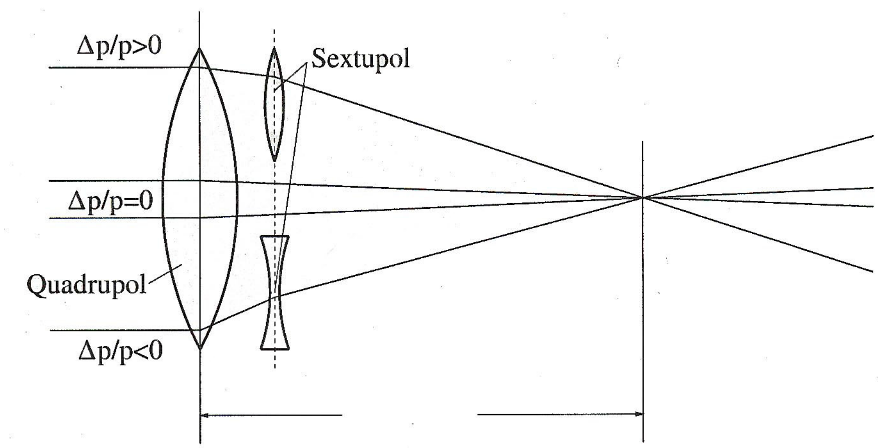
Focal length of the lens is dependent upon energy



Larger energy particles have longer focal lengths

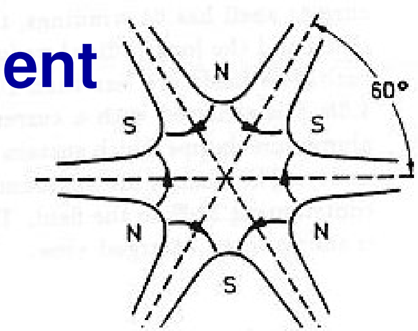
Chromatic Aberration Correction

By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent Quadrupole

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$



Chromatic Aberration Correction

Chromaticity, ν', is the change in the tune with energy

$$\nu' = \frac{d\nu}{d\delta}$$

Sextupoles can change the chromaticity

$$\Delta\nu_x' = \frac{1}{2\pi}\left(\Delta S\,\beta_x D_x\right)$$

$$\Delta\nu_y' = -\frac{1}{2\pi}\left(\Delta S\,\beta_y D_x\right)$$

where

$$\Delta S = \left(\partial^2 B_y \big/ \partial x^2\right) \cdot \text{length} \big/ (2B\rho)$$

Let's now turn on the RF cavity

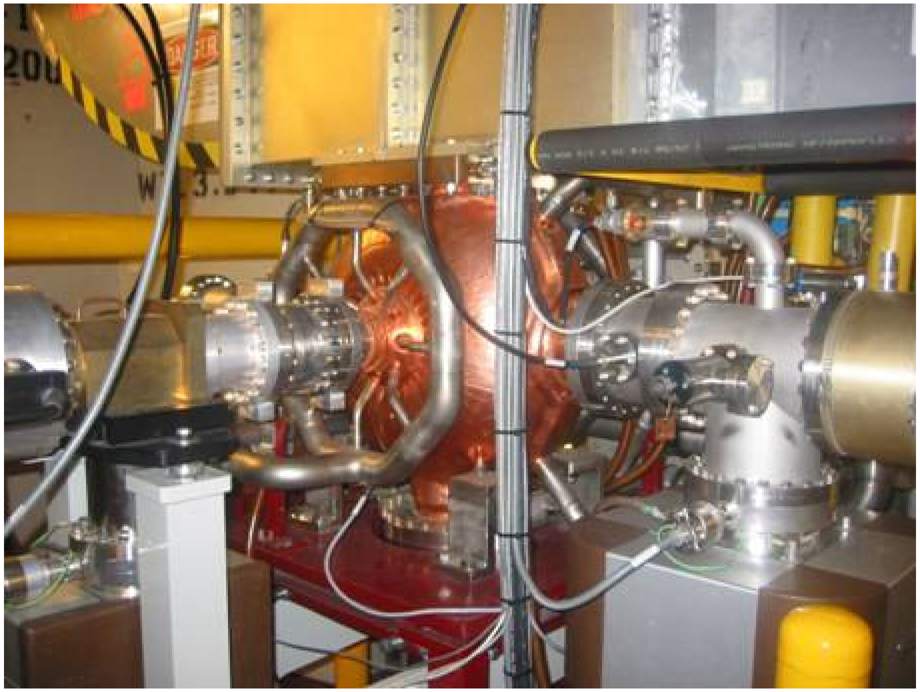
The longitudinal equations of motion become

$$\frac{d\phi}{dt} = -\alpha\omega_{RF}\delta \qquad\qquad \frac{d\delta}{dt} = \frac{eV_{RF}(t) - U(\delta)}{E_0 T_0}$$

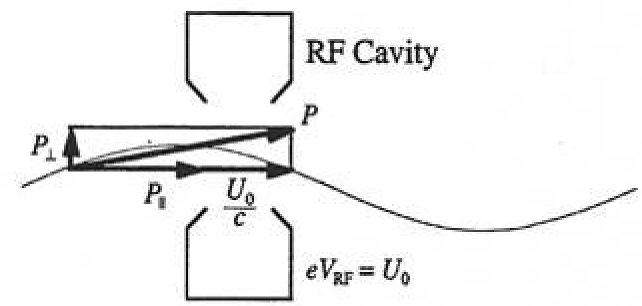
$\phi =$ **Phase of arrival at a fixed point along the closed orbit, in radians, at the RF frequency**

$\omega_{RF} = 2\pi f_{RF} =$ **Angular RF frequency**

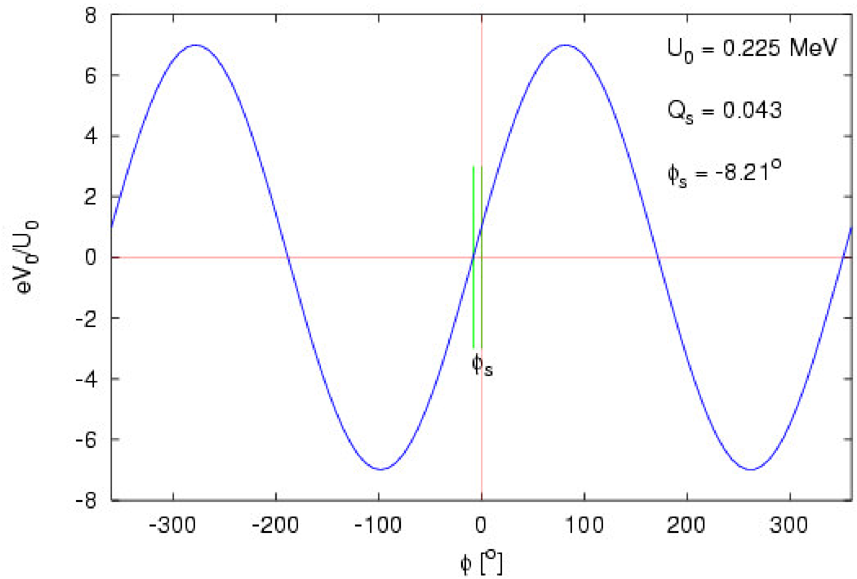
ALS example of RF cavity

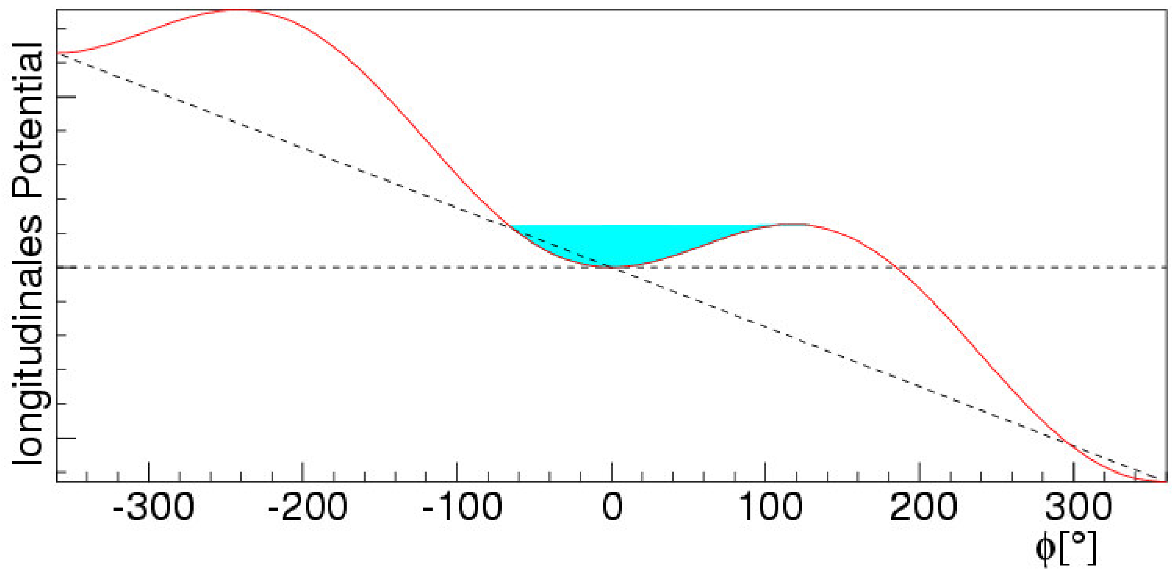


- Cavities replenish the energy loss due to synchrotron radiation



Synchrotron Oscillations





Synchrotron tune

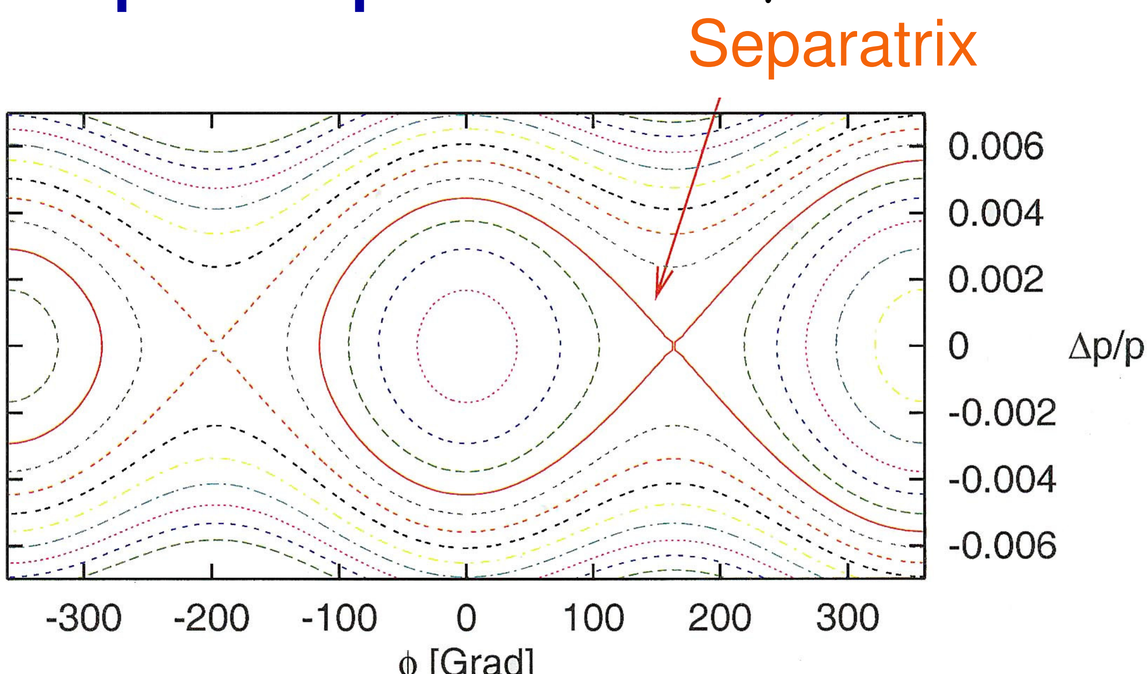
Solving the equations of motion, the synchrotron tune can be calculated

$$\Delta\ddot{E} + 2a_s \Delta\dot{E} + \Omega^2 \Delta E = 0$$

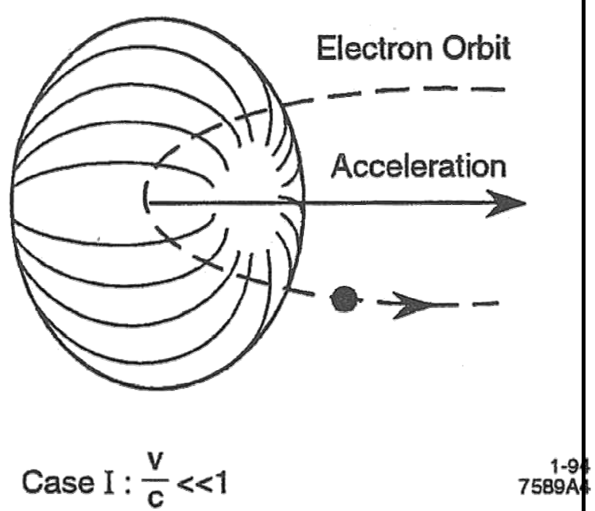
$$a_s = \frac{1}{2T_0}\frac{dU_0}{dE}$$

$$\Omega = \omega_0 \sqrt{-\frac{eV_{RF} q \cos\Psi_s}{2\pi\beta^2 E}\left(\alpha - \frac{1}{\gamma^2}\right)}$$

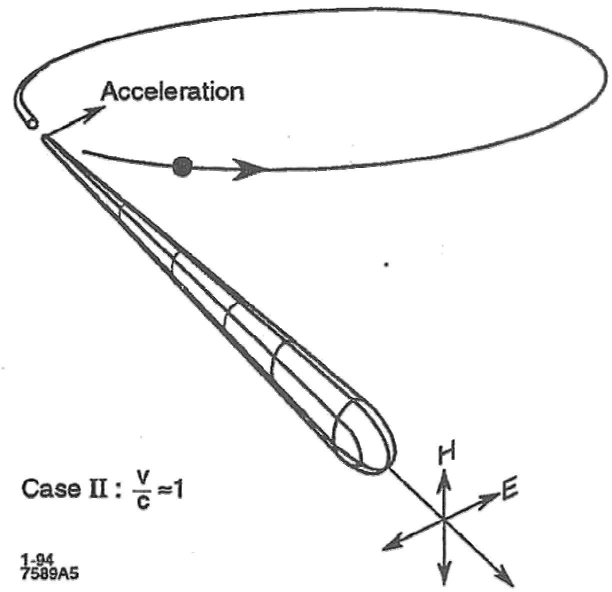
The longitudinal phase space



- Radiated power increases at higher velocities
- Radiation becomes more focused at higher velocities



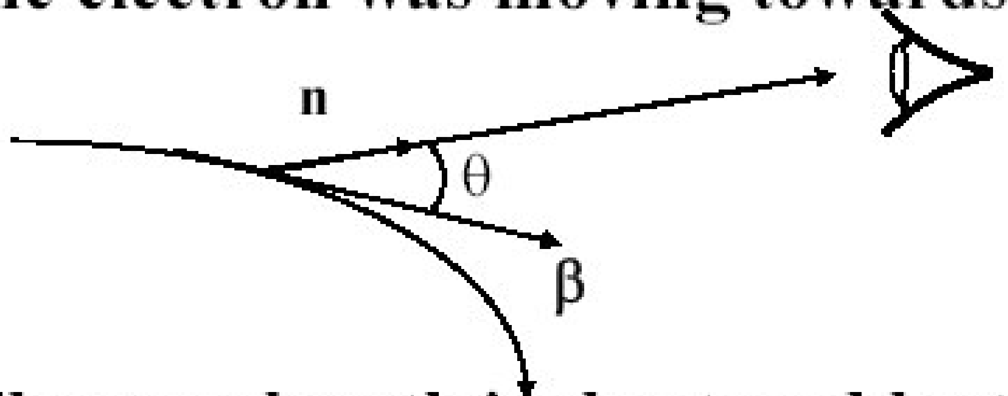
At low electron velocity (non-relativistic case) the radiation is emitted in a non-directional pattern



When the electron velocity approaches the velocity of light, the emission pattern is folded sharply forward. Also the radiated power goes up dramatically

Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) T_{emit}$$

The wavelength is shortened by the same factor

$$\lambda_{obs} = (1 - \beta \cos\theta)\, \lambda_{emit}$$

in ultra-relativistic case, looking along a tangent to the trajectory

$$\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}$$

since

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}$$

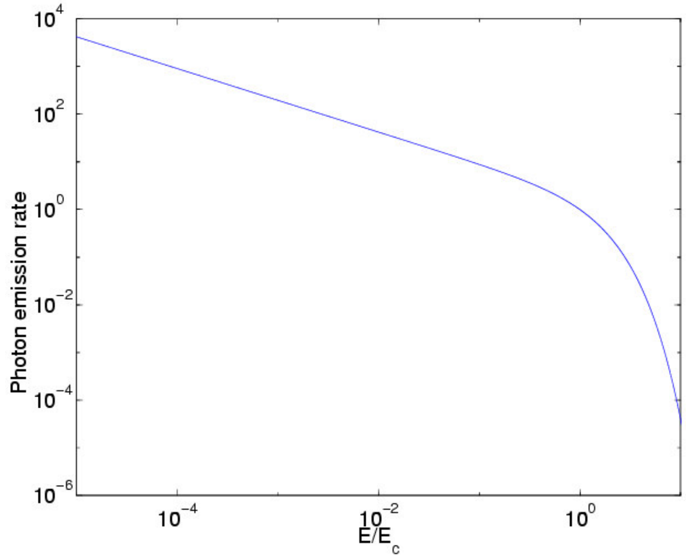
Radiation

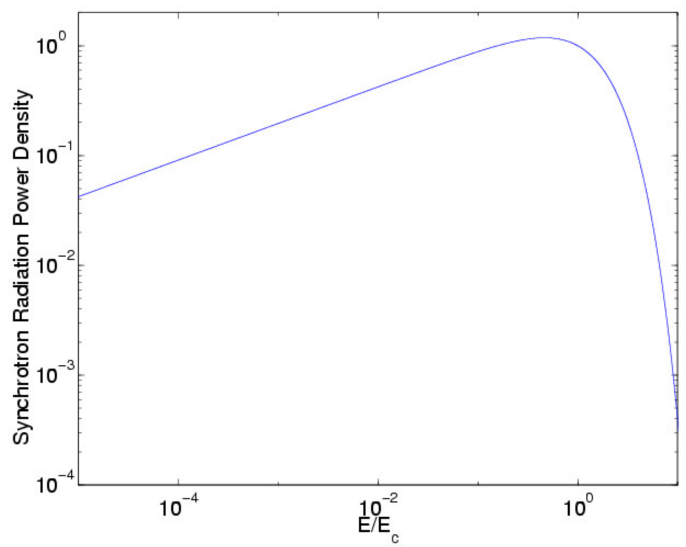
The power emitted by a particle is

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

and the energy loss in one turn is

$$U_0 = \frac{4\pi}{3}\alpha\hbar c \frac{\gamma^4}{\rho^2}$$





Radiation damping

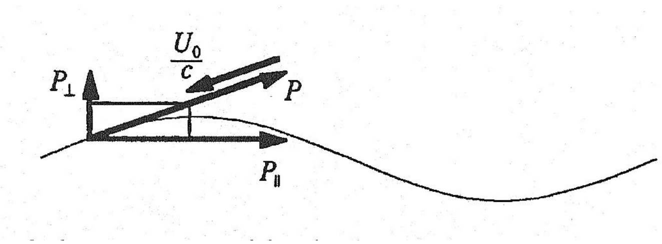
Energy damping:

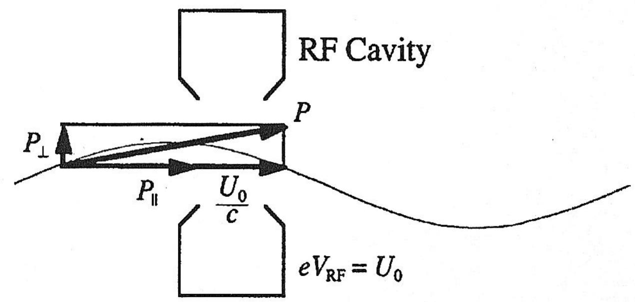
Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha\hbar c^2 \frac{\gamma^4}{\rho^2}$$

Transverse damping:

Energy loss is in the direction of motion while the restoration in the s direction





Quantum excitation

The synchrotron radiation emitted as photons, the typical photon energy is

$$u_c = \hbar\omega_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho}$$

The number of photons emitted is

$$N = \frac{4}{9}\alpha c\frac{\gamma}{\rho}$$

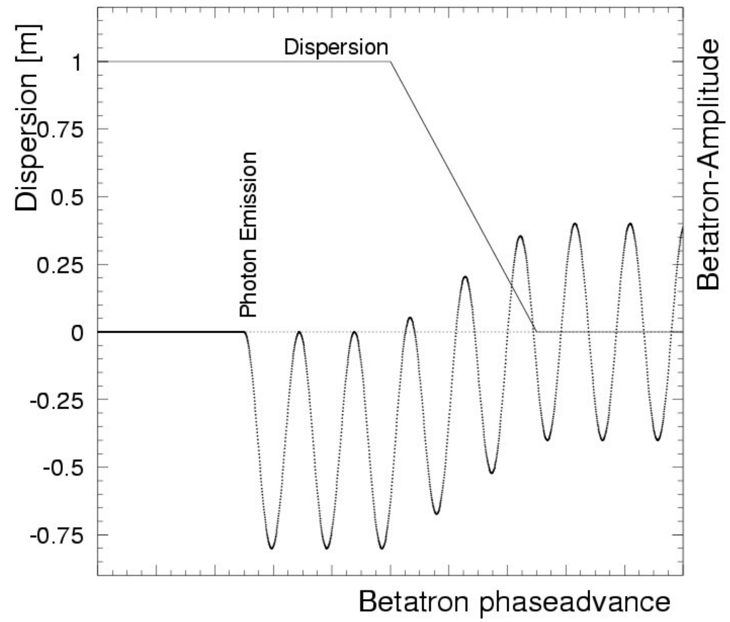
With a statistical uncertainty of $\sqrt{N}$

The equilibrium energy spread and bunch length is

$$\left(\frac{\sigma_e}{E}\right)^2 = 1.468\cdot 10^{-6}\frac{E^2}{J_\varepsilon \rho} \text{ and } \sigma_L = \frac{\alpha R}{f_0}\sigma_e$$

Emittance and beam size

Particles change their energy in a region of dispersion undergoes increase transverse oscillations. This balanced by damping gives the equilibrium emittances.



The beam size is then

$$\sigma_x = \sqrt{\beta_x \varepsilon + \left(D_x \frac{\sigma_e}{E} \right)^2}$$