USPAS 2012: Grand Rapids, MSU

Transverse and Longitudinal Beam Dynamics Fundamentals

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Outline

- Motivation
- Transverse Beam Dynamics
  – Hill's Equation
  – Twiss Functions (Beta-Function, …)
  – Tune, Resonances
  – Emittance
  – Matrix Formalism
    - Basis for simulation codes
- Longitudinal Dynamics
  – Time of Flight, Synchrotron Oscillations
- Radiation
  – Damping/Excitation, Equilibrium Emittances

http://als.lbl.gov/als_physics/csteier/uspas12/

Motivation

- This course will deal in detail with measurements involving many areas of transverse (and longitudinal) single (and multiple) particle dynamics
- Most (but not all) of you already have learned all fundamentals
- Still would like to remind you of all concepts, to get all to somewhat consistent starting point for class
  - For transverse dynamics will introduce lattice functions in two different ways (including the one usually used in lattice codes, which you might not have learned, yet).

- *No need to get scared by this pretty dense lecture. The remainder of the week will be much more practical – and does not require that you completely understand everything in this recap*

- *Disclaimer: Our class is storage ring biased. Basic concepts and measurements are applicable to transfer lines and linacs, but details are different. If you have questions regarding lines, linacs, protons: You are welcome to ask at any time.*

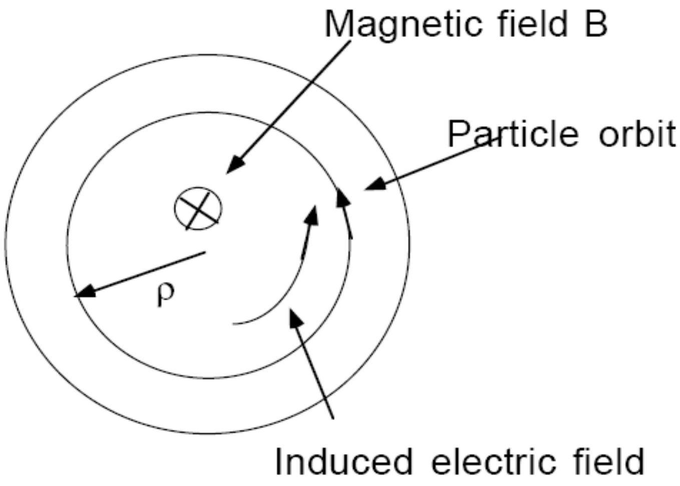
Transverse Beam-dynamics: Terminology

- Linear beamdynamics (today) determined by:
  - Dipoles
  - Quadrupoles (lenses)
  - Solenoids
  - rf-resonators
  - (synchrotron radiation)
- Nonlinear (Thursday):
  - Sextupoles, higher multipoles, errors, insertion devices (undulators/wigglers), stochastic nature of SR, …
- Trajectory/Orbit – (single pass/periodic)
  - Closed orbit: closed, periodic trajectory around a ring (closes after one turn in position and angle).
  - Particles that deviate from the closed orbit will oscillate about it (transverse: Betatron oscillations, longitudinal: Synchrotron Oscillations)

Induction Accelerators

$$\nabla \times \overline{E} = -\frac{d\overline{B}}{dt}$$

Faraday's law

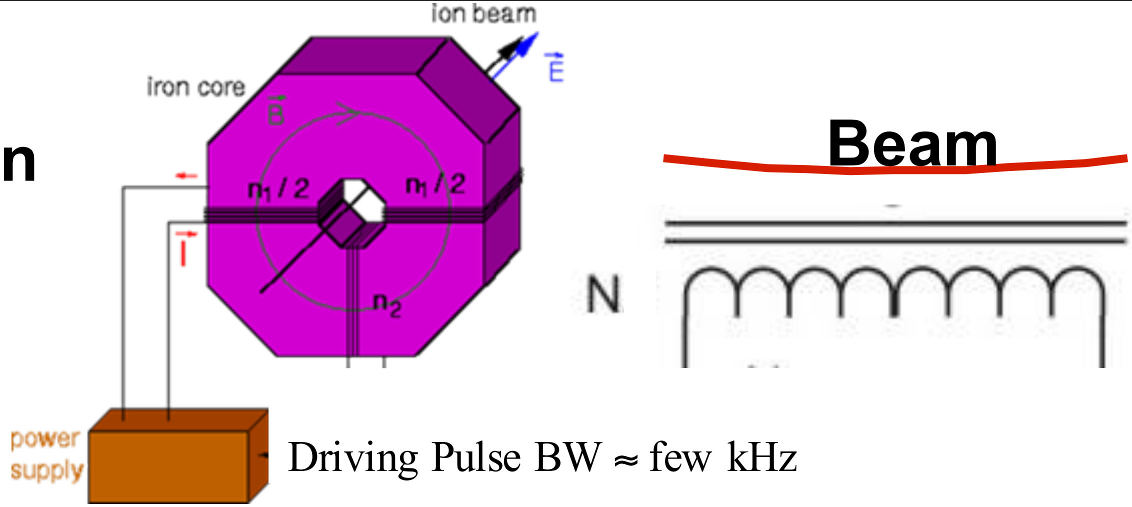


Betatron (Kerst 1940)

- For electrons (reach relativistic velocities quickly) one can use pulsed machines
- Acceleration provided by induced voltage due to change in magnetic field
- Geometry determines synchronicity

$$\Delta\phi = 2\pi\rho^2 B_{\max}$$

Induction Linac (1964)

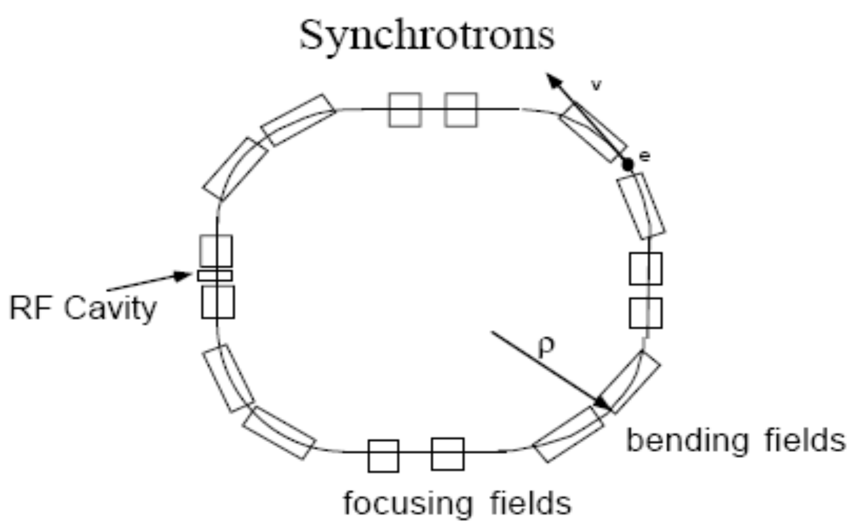


Driving Pulse BW ≈ few kHz

Induction accelerators can be very efficient (>50%) and allow for very high currents (~ 1kA) at relatively moderate energies (few MeV)

Synchrotron (1945)

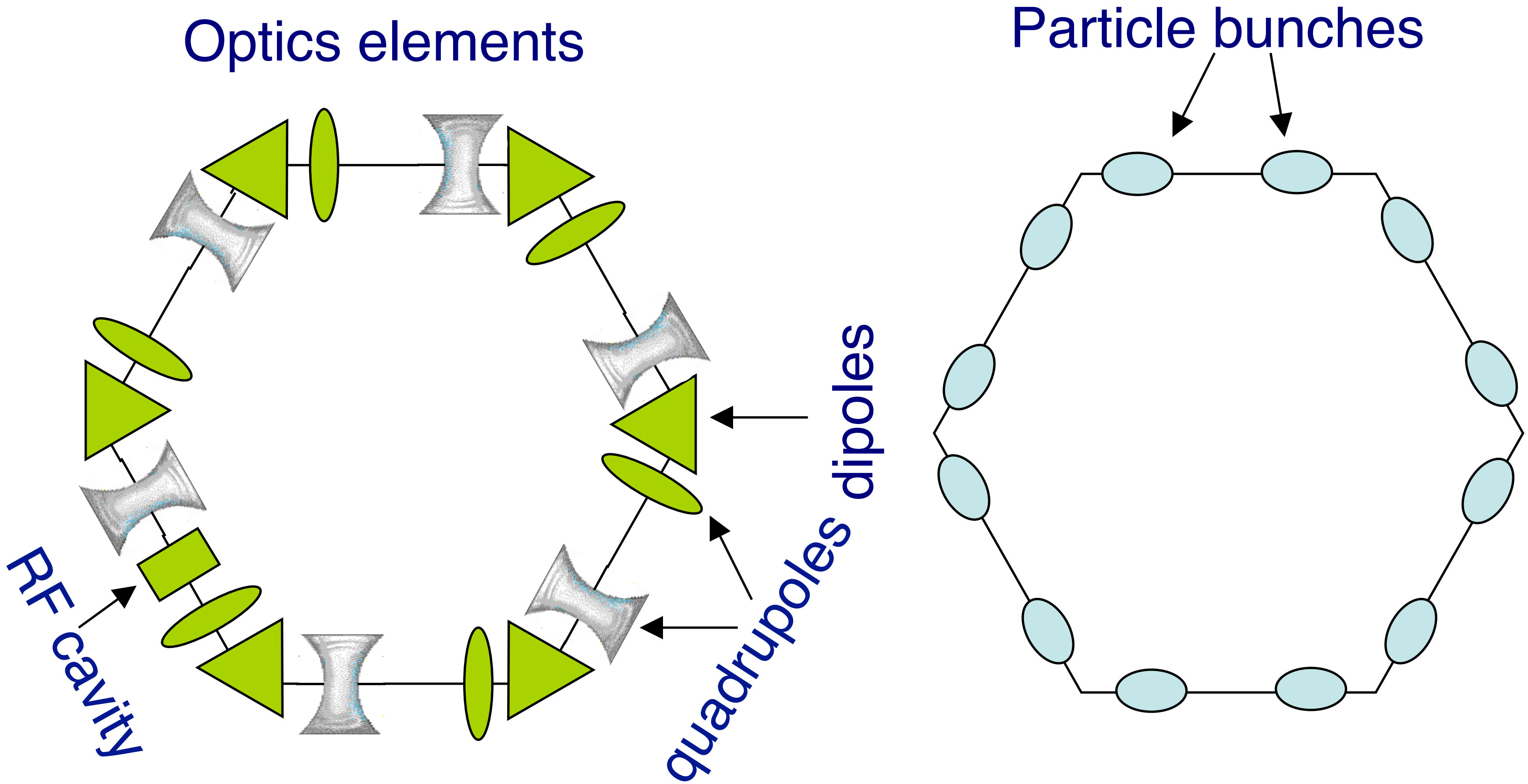
- Synchrotrons as well as the linear accelerators (linacs) mentioned before, are important in elementary particle physics research, where highest possible particle energies are needed.
- A synchrotron is a circular accelerator which has one (or a few) electromagnetic resonant cavity to accelerate the particles. A constant orbit is maintained during the acceleration.
  - First ones were weak focusing (very large vacuum chambers and magnets)
  - Later strong focusing.
- Originally ramping/cycling, today often storage rings (many h)



The synchrotron concept seems to have been first proposed in 1943 by the Australian physicist Mark Oliphant.

Synchrotrons / Storage Rings

In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.



Equations of Motion in a Storage Ring

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- **Lorentz Force**

$$F = ma = e(E + v \times B),$$

m **is the relativistic mass of the particle,**

e **is the charge of the particle,**

v **is the velocity of the particle,**

a **is the acceleration of the particle,**

E **is the electric field and,**

B **is the magnetic field.**

Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

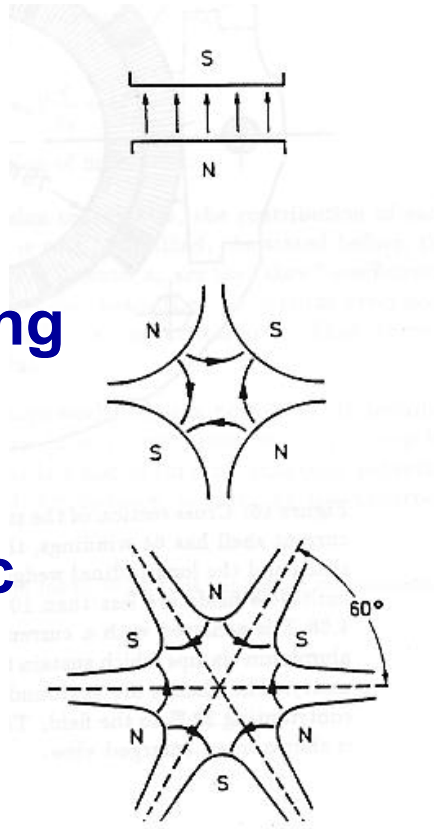
$$B_x = 0$$
$$B_y = B_o$$

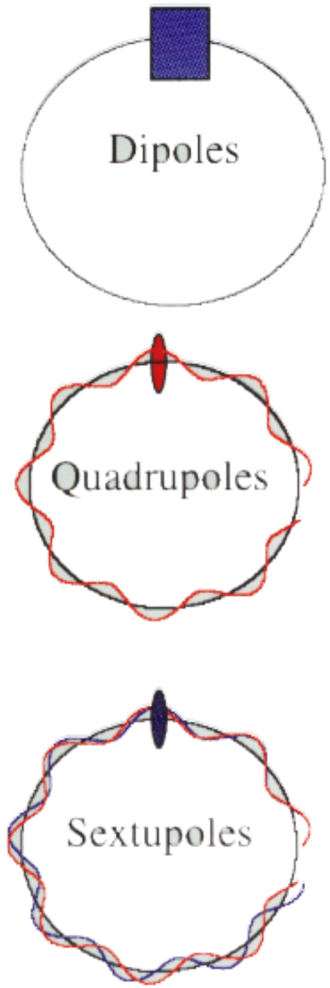
Quadrupoles → used for focussing

$$B_x = Ky$$
$$B_y = -Kx$$

Sextupoles → used for chromatic correction

$$B_x = 2Sxy$$
$$B_y = S(x^2 - y^2)$$

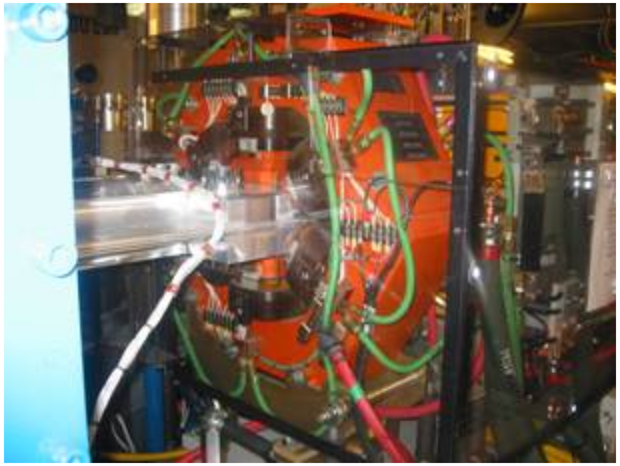




Practical Magnet Examples at the ALS

Dipoles

Quadrupoles



Sextupoles

Differential Equation/Matrix Formalism

There are two approaches to introduce the motion of particles in a storage ring

1. **The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, …**

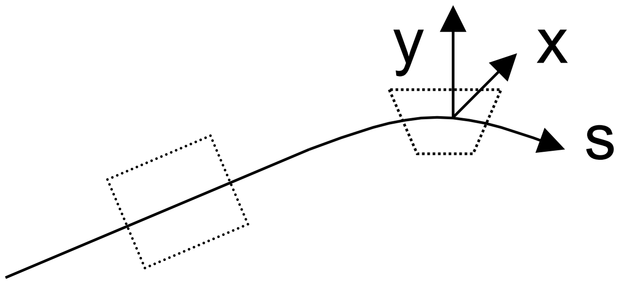
2. **The way that our computer models actually do it**

I will begin with the first way (as a brief recap) but spend most of the time with the second approach

Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



Typically the coordinate system chosen is the one that allows the easiest field representation

First approach – traditional one

This approach (differential equations) provides some insights into concepts but is limited in usefulness for actual calculations

We begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2x}{ds^2} = -k(s)x,$$

where $k = \frac{B_T}{(B\rho)a}$, **with**

B_T **being the pole tip field**

a **the pole-tip radius, and**

$B\rho[\text{T-m}] \approx 3.356\, p[\text{GeV/c}]$

ALS

Hills equation

The solution can be parameterized by a pseudo-harmonic oscillation of the form

$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s)+\varphi_0)$$

$$x_\beta'(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s)+\varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s)+\varphi_0)$$

where $\beta(s)$ is the beta function,

$\alpha(s)$ is the alpha function,

$\varphi_{x,y}(s)$ is the betatron phase, and

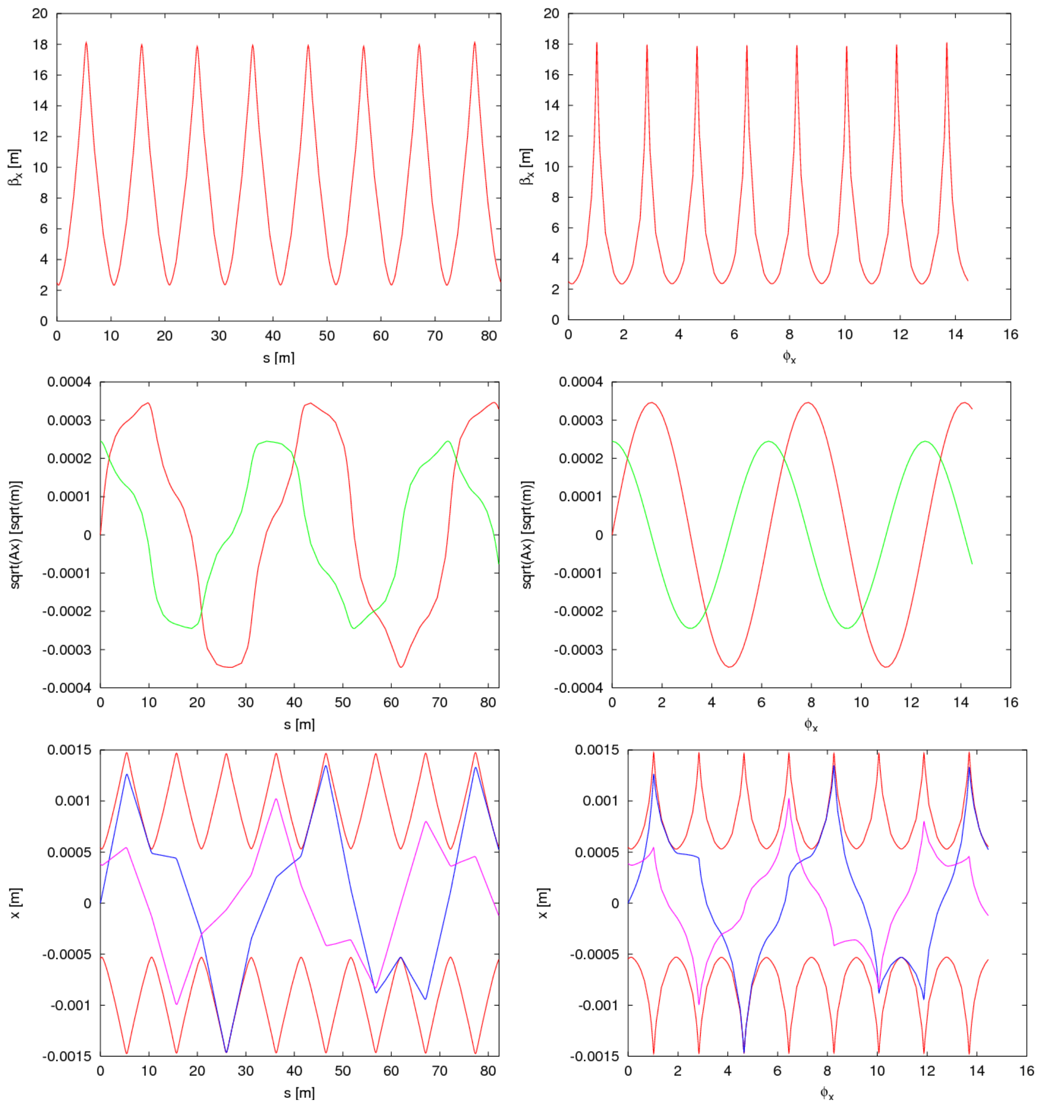
ε is an action variable

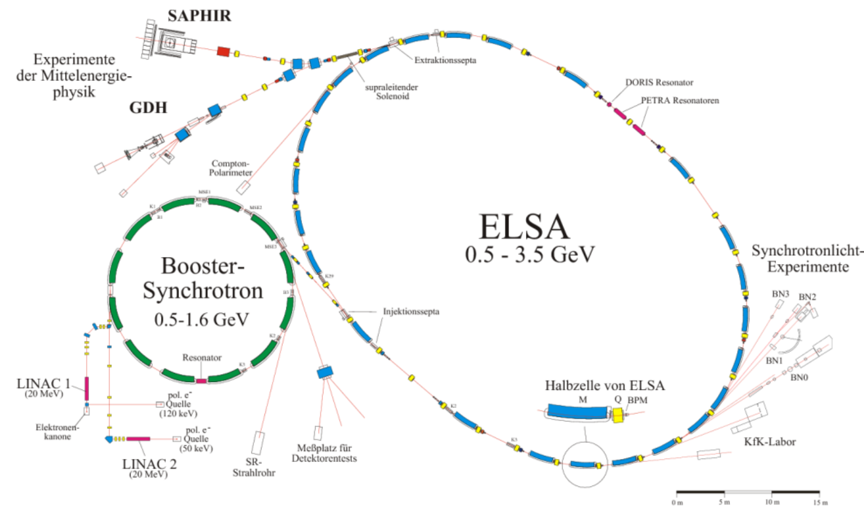
$$\alpha = -\frac{\beta'}{2},$$

$$\varphi = \int_0^s \frac{ds}{\beta}$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

Example of Twiss parameters and trajectories





ELSA (Electron Stretcher and Accelerator) in Bonn is an example of a relatively simple FODO lattice

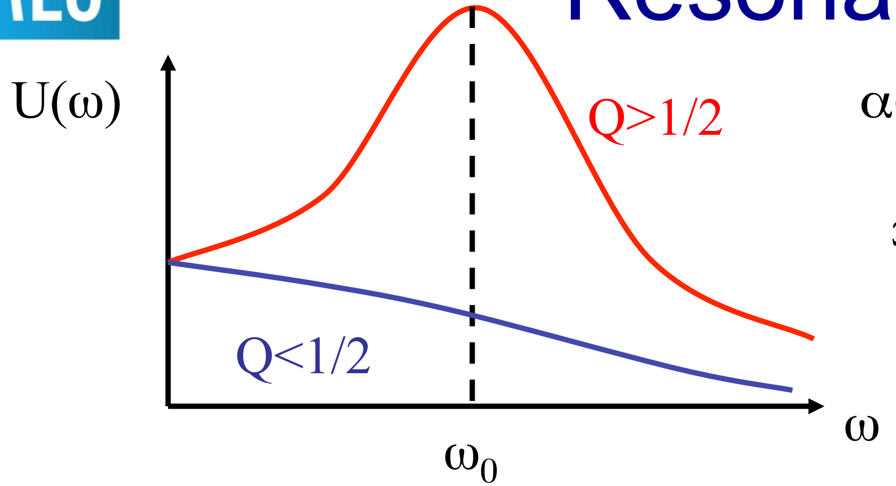
- **Beta Function highly periodic**
- **Trajectories in real space are piecewise straight (with deflections at quadrupoles)**
- **If one transforms with beta functions and phase advance, they start to look like harmonic functions (sine/cosine)**

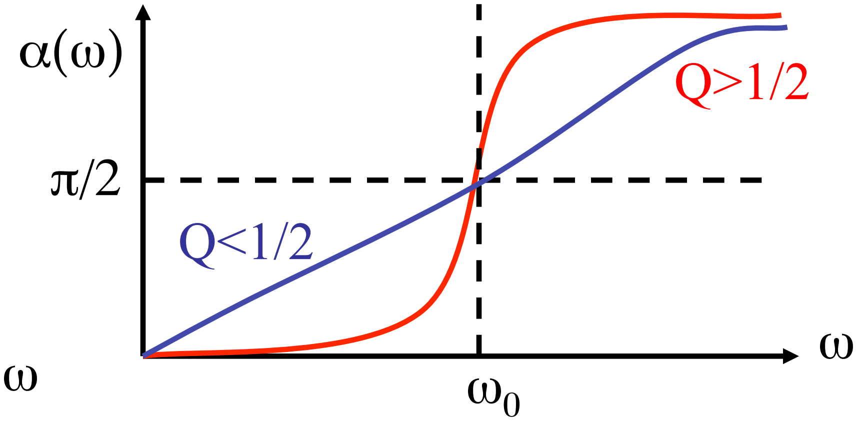
Damped and driven harmonic oscillator – Resonances (will come back Thursday)

- The general solution is a sum of a transient (the solution for damped undriven harmonic oscillator, homogeneous ODE) that depends on initial conditions, and a steady state (particular solution of the nonhomogenous ODE) that is independent of initial conditions and depends only on driving frequency, driving force, restoring force, damping force, Damped harmonic oscillator:

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q}\frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m}\cos(\omega t)$$

Resonance effect





$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

Excitation at the Eigenfrequencies by strong wind

2nd Approach: How to calculate particle trajectories and lattice functions

Begin with equations of motion → Lorentz force

↓

Change dependent variable from time to longitudinal position

↓

Integrate particle trajectory around the ring and find the closed orbit

↓

Generate a map around the closed orbit

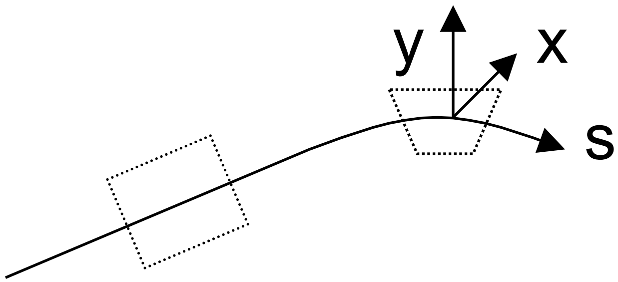
↓

Analyze and track the map around the ring

Coordinate System

Change dependent variable from time to longitudinal position, *s*

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



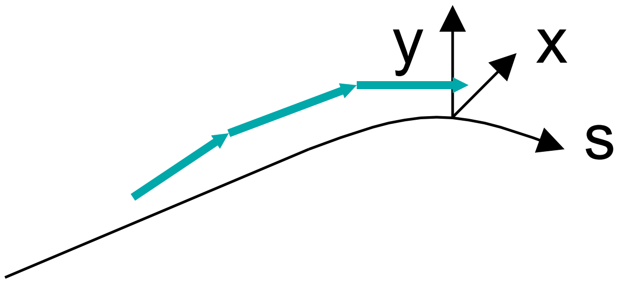
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

$$x,\quad x' = \frac{dx}{ds},\quad y,\quad y' = \frac{dy}{ds},\quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$

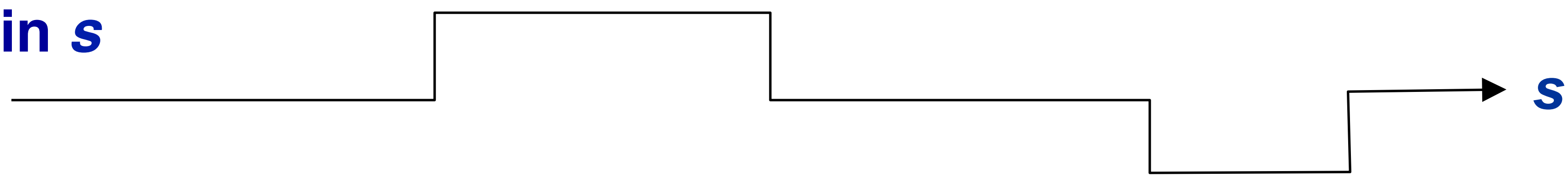


Note sometimes one uses canonical momentum rather than x' and y'

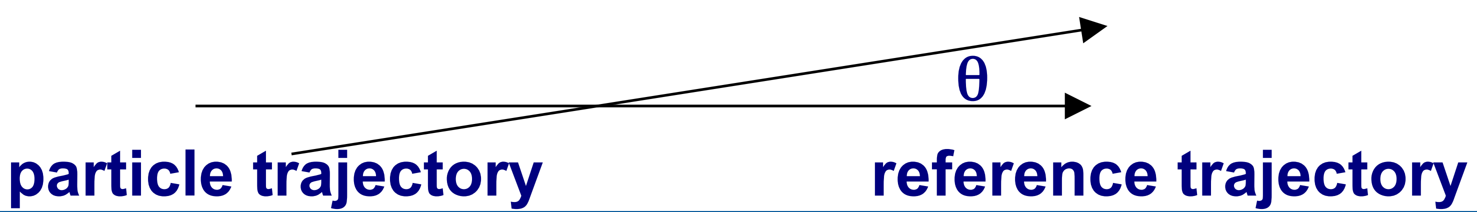
Approximation

Everything up to now there was general. No discussion of the field representation or the integrator. In many codes simplifications are made.

1. **The velocity of the particle is the speed of light → $v = c$**
2. **The magnetic field is isomagnetic. Piecewise constant in s**

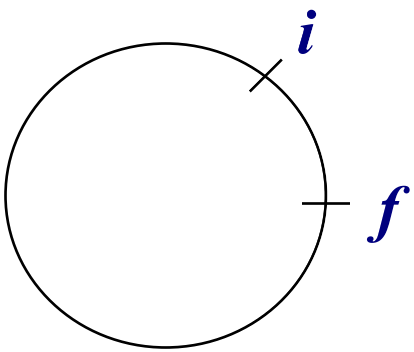


3. **The angle of the particles with respect to the reference particle is small and can assume that $\theta = \tan\theta$**



Transfer Matrix

One can write the linear transformation between one point in the storage ring (i) to another point (f) as



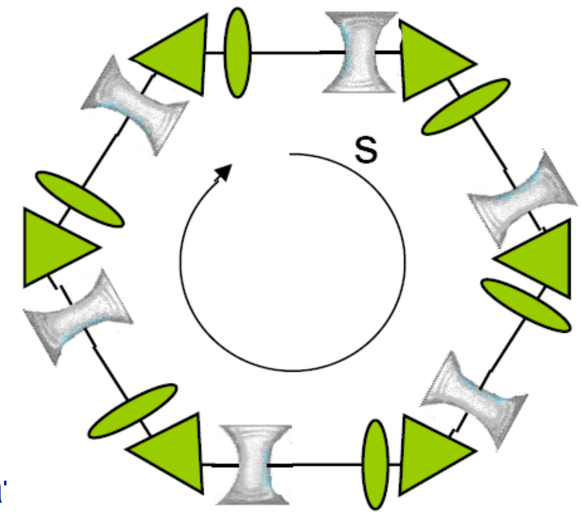
$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

this is for the case of uncoupled horizontal motion. One can extend this to 4x4 or 6x6 cases.

Piecewise constant magnetic fields

- **General** transfer matrix **from s_0 to s**

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0)\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix}\begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$



- **Note that**

$$\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$$

which is always true for conservative systems

- **Note also that** $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- **The accelerator can be build by a series of matrix multiplica**

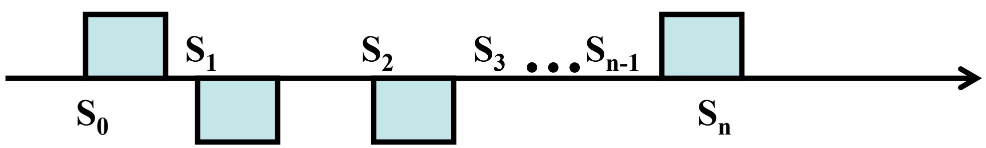
$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1})\ldots\mathcal{M}(s_3|s_2)\cdot\mathcal{M}(s_2|s_1)\cdot\mathcal{M}(s_1|s0)$$

from $\mathbf{s_0}$ to $\mathbf{s_1}$

from $\mathbf{s_0}$ to $\mathbf{s_2}$

from $\mathbf{s_0}$ to $\mathbf{s_3}$

from $\mathbf{s_0}$ to $\mathbf{s_n}$



Examples of transfer matrices

Drift of length L

$$R_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

The matrix for a focusing quadrupole of gradient $k = (\partial B / \partial x)/(B\rho)$ and of length l_q

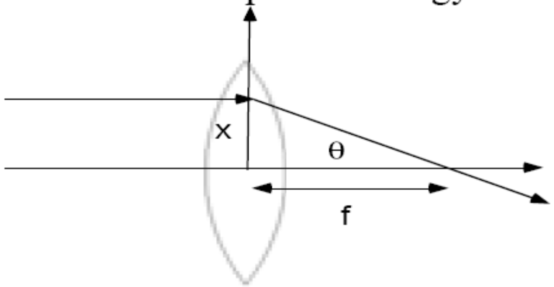
$$R_{Quad} = \begin{pmatrix} \cos\phi & \sin\phi / \sqrt{|k|} \\ -\sqrt{|k|}\sin\phi & \cos\phi \end{pmatrix}$$

The matrix for a zero length thin quadrupole $K = |k|\, l_q$

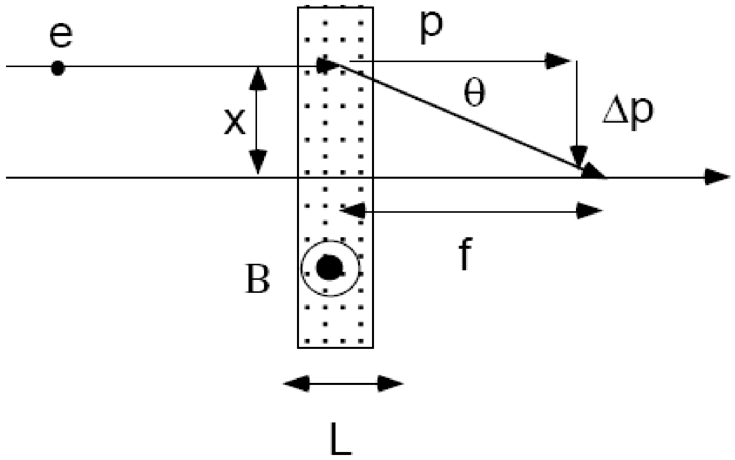
$$R_{thin-lens} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}$$

Magnetic lenses: Quadrupoles

Magnetic focusing fields:
Optical analogy: Thin lens, focal length f

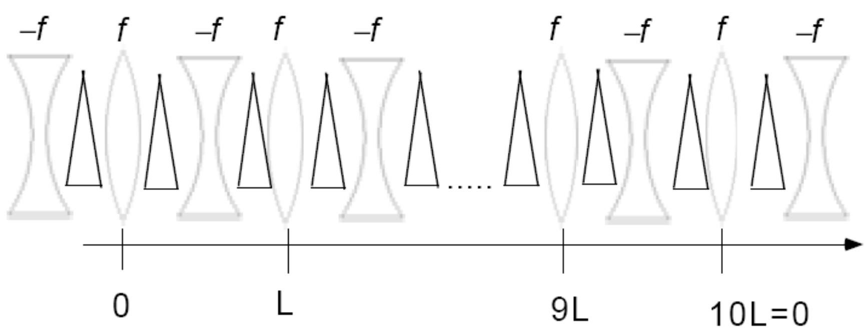


$$\theta = \frac{x}{f}$$



- Thin lens representation

FODO cell

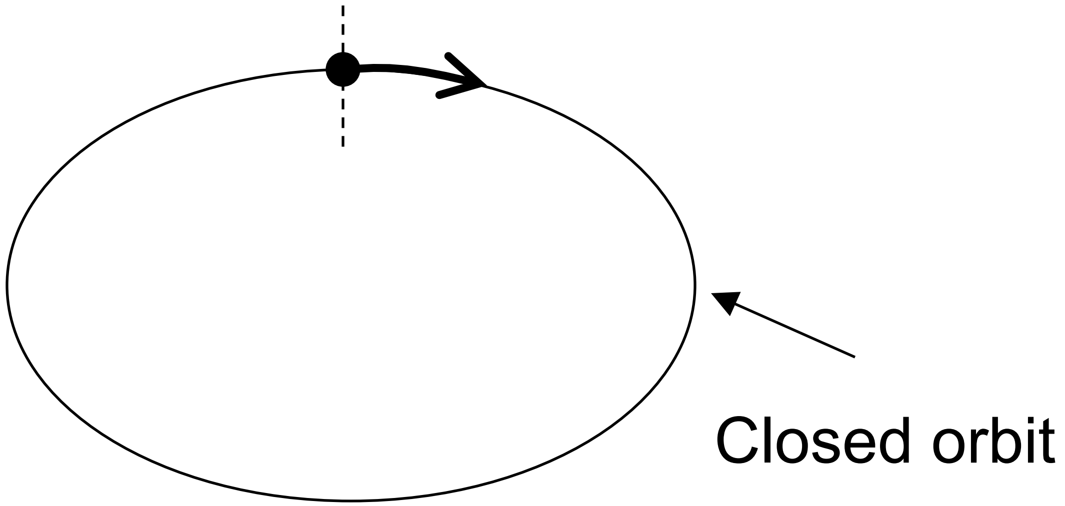


$$\begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix} \quad \begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix}$$

Drift: Thin lens:

Find the Closed Orbit

A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.



In every working story ring there exists at least one closed orbit.

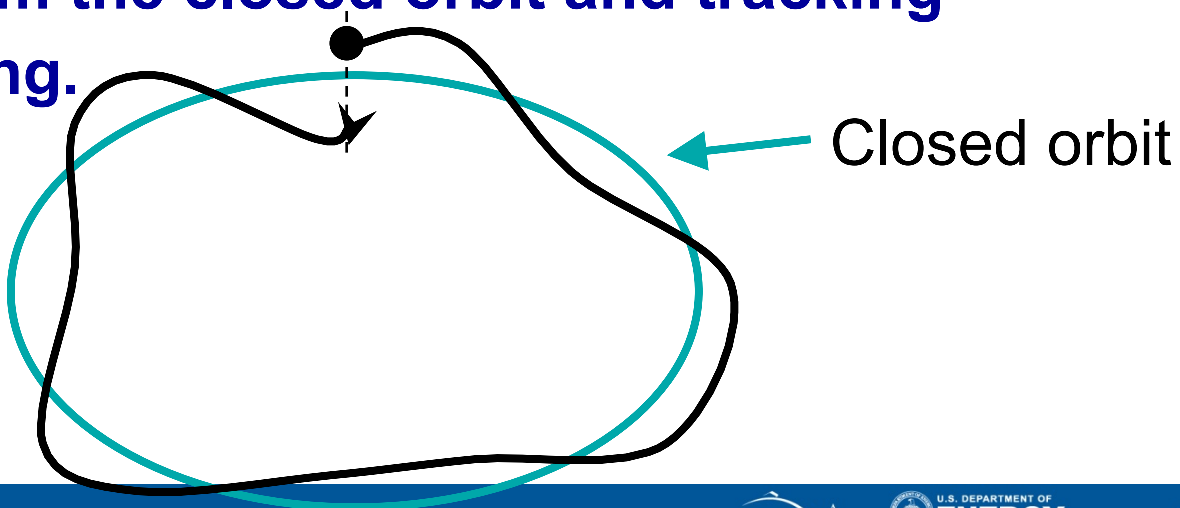
Generate a one-turn Map Around the Closed Orbit

A one-turn map, *R*, maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

$$x_f = x_i + \frac{dx_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

$$x'_f = x'_i + \frac{dx'_f}{dx_i}\left(x_i - x_{i,co}\right) + \frac{dx'_f}{dx'_i}\left(x'_i - x'_{i,co}\right) + \ldots$$

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



Computation of beta-functions and tunes

The one turn matrix (the first order term of the map) can be written

$$R_{one-turn} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\varphi + \alpha\sin\varphi & \beta\sin\varphi \\ -\gamma\sin\phi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

Where α, β, γ are called the Twiss parameters

$$\alpha = -\frac{\beta'}{2},$$

and the betatron tune, $\nu = \phi/(2*\pi)$

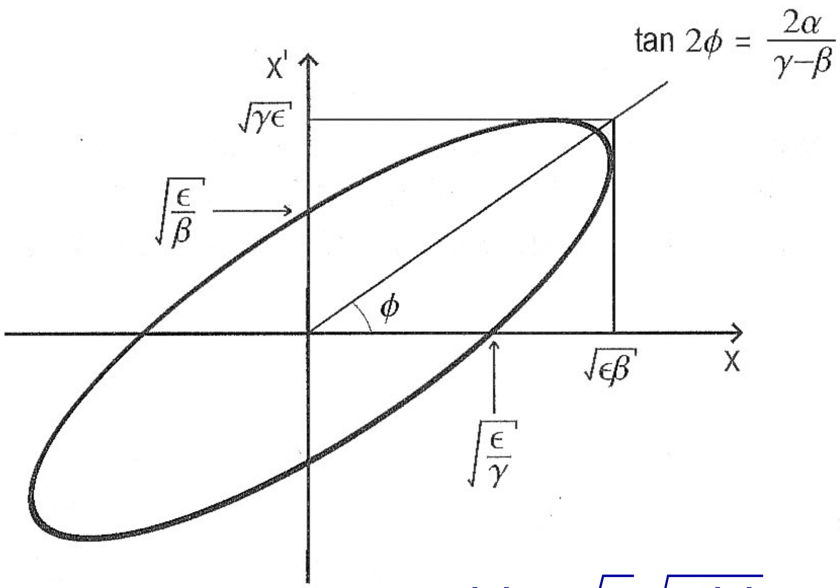
$$\gamma = \frac{1+\alpha^2}{\beta}$$

For long term stability ϕ is real →

$|TR(R)| = |2\cos\phi| < 2$

Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ε:

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

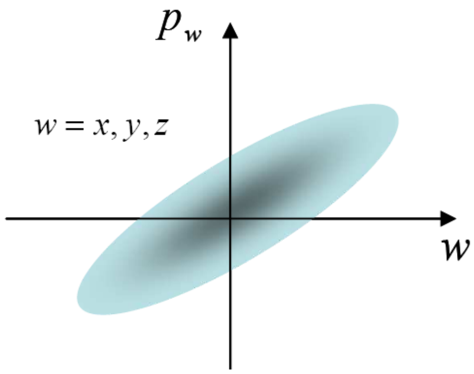
$$x_\beta(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos(\varphi(s) + \varphi_0)$$

$$x'_\beta(s) = -\sqrt{\varepsilon}\frac{\alpha}{\sqrt{\beta(s)}}\cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\sin(\varphi(s) + \varphi_0)$$

Emittance Definition

- Consider the decoupled case and use the {*w*, *w*} plane where *w* can be either *x* or *y*:
  - The emittance is the phase space area occupied by the system of particles, divided by π

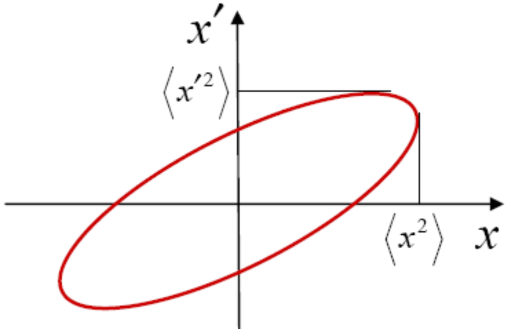
$$\varepsilon_w = \frac{A_{ww'}}{\pi} \qquad w = x, y$$



- *x'* and *y'* are conjugate to *x* and *y* when $B_z = 0$ and in absence of acceleration. In this case, we can immediately apply the Liouville theorem:
  - For such a system the emittance is an invariant of the motion.
- This specific case is very common in accelerators:
  - For most of the elements in a beam transferline, such as dipoles, quadrupoles, sextupoles, …, the above conditions apply and the emittance is conserved.

Emittance Definition/Statistical

- Emittance defined as the phase space area occupied by an ensemble of particles
- Example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).
- Emittance can be defined as a statistical quantity (beam is composed of finite number of particles)



$$\varepsilon_{geometric,rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\langle x^2\rangle = \frac{\sum_{n=1}^{N} x_n^2}{N} \cong \frac{\int x^2 f_{2D}(x,x')\,dx\,dx'}{\int f_{2D}(x,x')\,dx\,dx'}$$

This is equivalent to associate to the real beam an *equivalent or phase ellipse* in the phase space with area $\pi\varepsilon_{rms}$ and equation:

$$\frac{\langle x'^2\rangle}{\varepsilon_{rms}} x^2 + \frac{\langle x^2\rangle}{\varepsilon_{rms}} x'^2 - 2\frac{\langle x\,x'\rangle}{\varepsilon_{rms}} x\,x' = \varepsilon_{rms}$$

Transport of the beam ellipse

Beam ellipse matrix

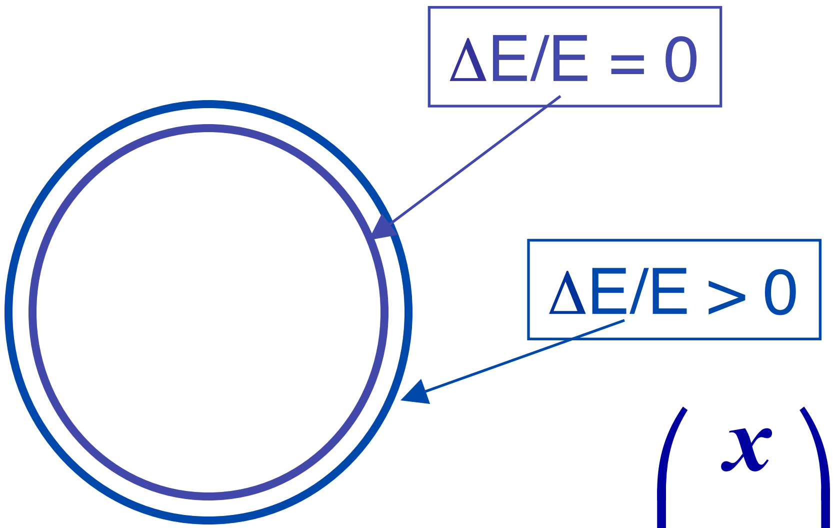
$$\sum_{beam}^{x} = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum_{beam,f}^{x} = R_{x,i-f} \sum_{beam,i}^{x} R_{x,i-f}^{T}$$

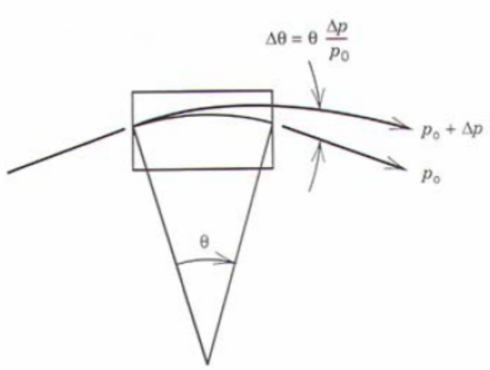
Off-Energy: Dispersion

Dispersion, *D*, is the change in closed orbit as a function of energy



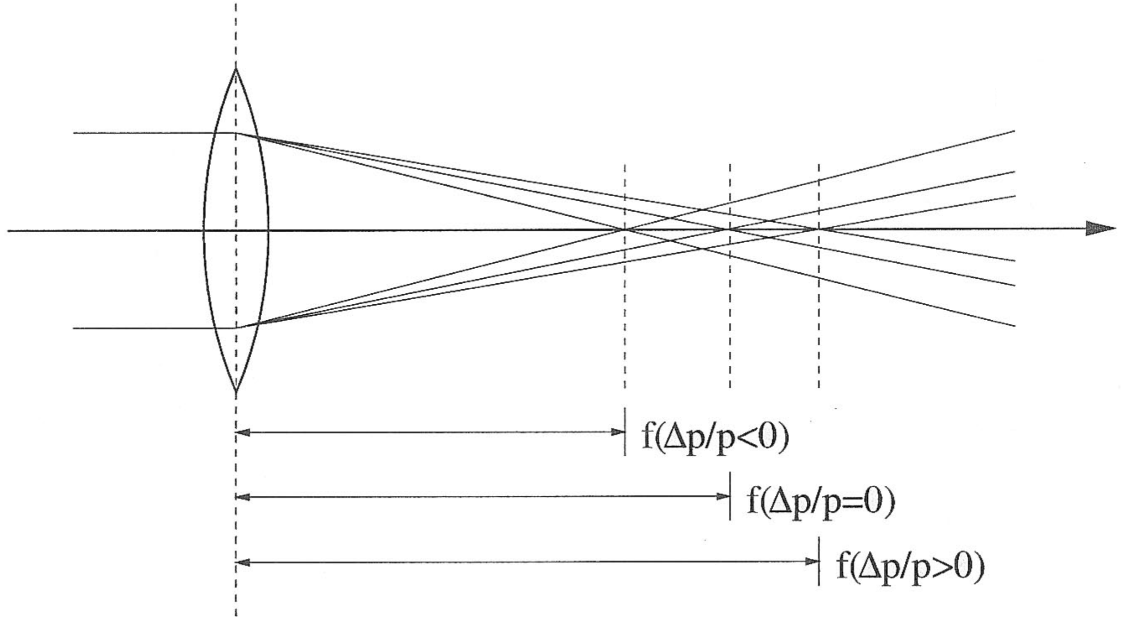
$$x = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix}_i$$



What happens off-energy ? Chromatic Aberration

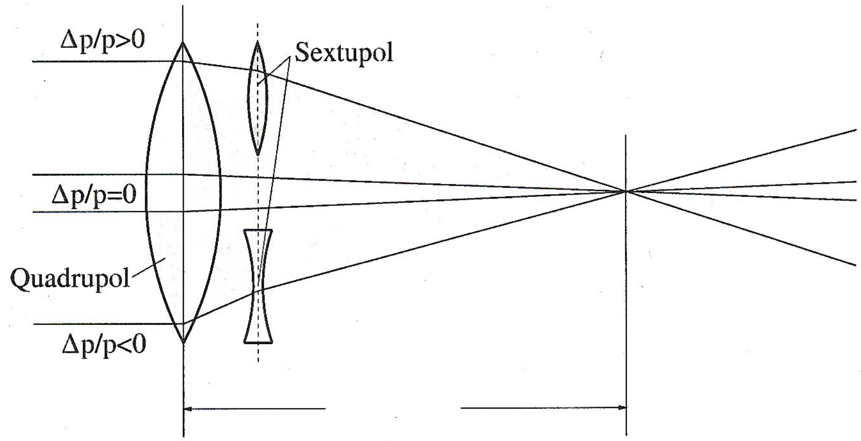
Focal length of the lens is dependent upon energy



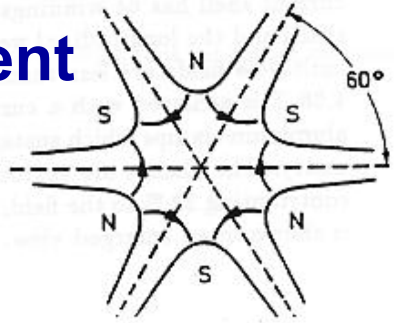
Larger energy particles have longer focal lengths

Chromatic Aberration Correction

By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent Quadrupole



$$B_x = 2Sxy$$

$$B_y = S(x^2 - y^2)$$

Chromatic Aberration Correction

Chromaticity, ν', is the change in the tune with energy

$$\nu' = \frac{d\nu}{d\delta}$$

Sextupoles can change the chromaticity

$$\Delta\nu_x' = \frac{1}{2\pi}\left(\Delta S \beta_x D_x\right)$$

$$\Delta\nu_y' = -\frac{1}{2\pi}\left(\Delta S \beta_y D_x\right)$$

where

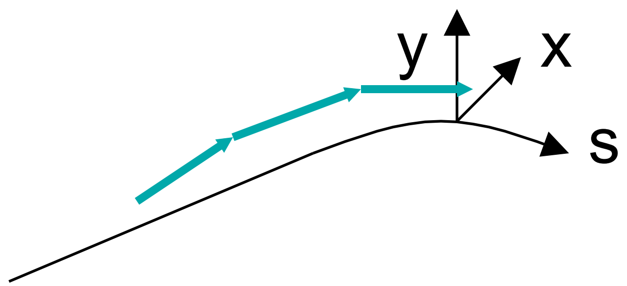
$$\Delta S = \left(\partial^2 B_y \big/ \partial x^2\right) \text{length} \Big/ \left(2B\rho\right)$$

Now to longitudinal Motion: Integrate – Recap from Before

Integrate through the elements – longitudinally same way as transversely

Use the following coordinates*

$$x,\quad x' = \frac{dx}{ds},\quad y,\quad y' = \frac{dy}{ds},\quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$



Note sometimes one uses canonical momentum rather than x' and y'

Examples of Element Transfer Matrix

Drift

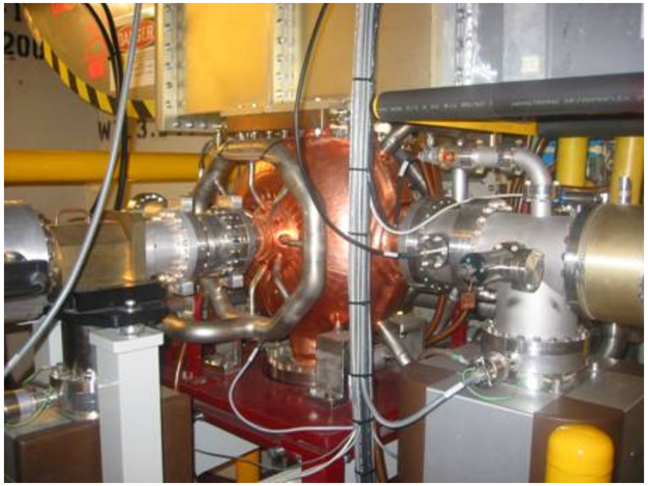
$$\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta^2\gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

thin RF cavity

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\omega\frac{e\hat{V}}{pc}\cos\phi & 1 \end{pmatrix}$$

coordinate vector

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ ct \\ \frac{\Delta p}{p} \end{pmatrix}$$

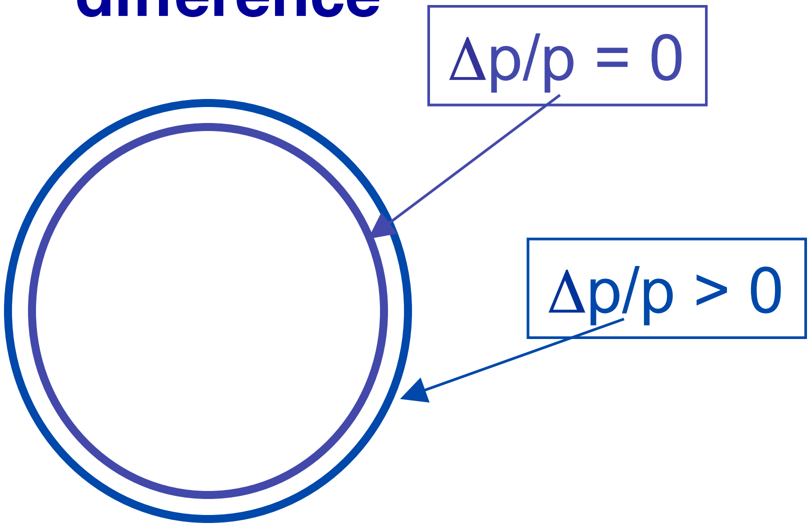


In addition there is path length effect in dipole – However, dipole transfer map is pretty confusing, so I will not write it down here … (compare slide 33).

How long does it take to complete revolution?

Assume that the energy is fixed → no cavity or damping

- **Find the closed orbit for a particle with slightly different energy than the nominal particle. The dispersion is the difference in closed orbit between them normalized by the relative momentum difference**

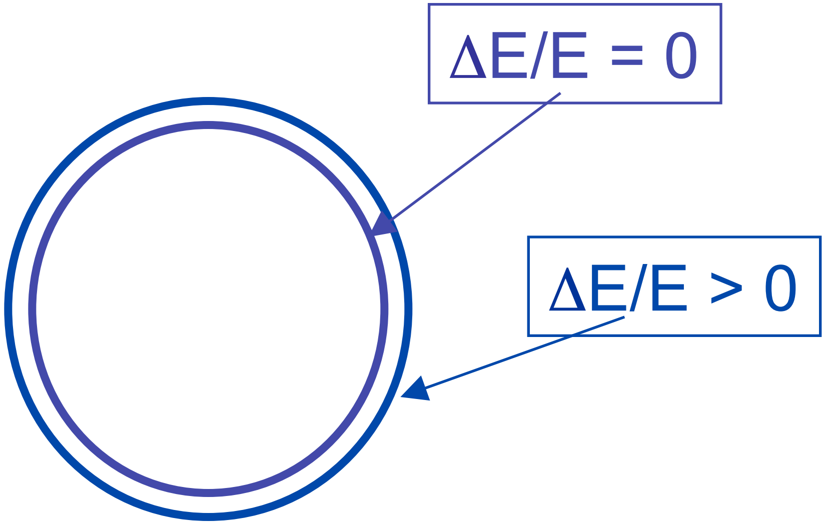


$$x = D_x \frac{\Delta p}{p}, y = D_y \frac{\Delta p}{p}$$

$$x' = D'_x \frac{\Delta p}{p}, y' = D'_y \frac{\Delta p}{p}$$

Momentum Compaction Factor

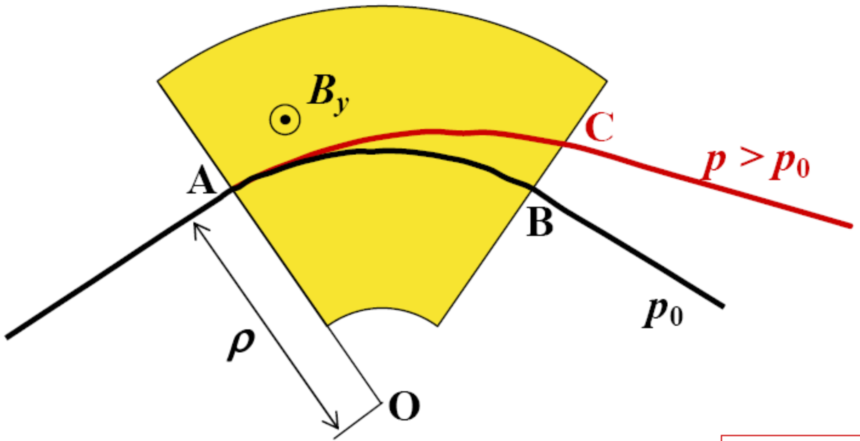
Momentum compaction, α, is the change in the closed orbit length as a function of momentum.



$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

$$\alpha = \int_0^{L_0} \frac{D_x}{\rho} ds$$

Momentum compaction element-by-element



$$\rho = \frac{p}{qB_z} = \frac{\beta\gamma m_0 c}{q B_z}$$

L_0 = Trajectory length between A and B

L = Trajectory length between A and C

$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0} \Rightarrow \boxed{\frac{\Delta L}{L_0} = \alpha_C \frac{\Delta p}{p_0}}$$

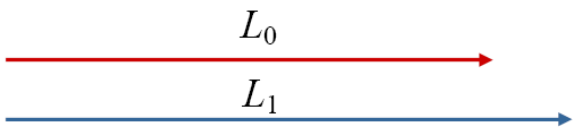
where α_C is constant

- In the example (sector bending magnet) $L > L_0$ so that $\alpha_C > 0$. Higher energy particles will leave the magnet later.

Ballistic time-of-flight

- Consider two particles with different momentum on parallel trajectories:

$$p_1 = p_0 + \Delta p$$



- At a given time *t*:

$$L_1 = (\beta_0 + \Delta\beta)ct \qquad L_0 = \beta_0\, ct \qquad \Rightarrow \frac{\Delta L}{L_0} = \frac{L_1 - L_0}{L_0} = \frac{\Delta\beta}{\beta_0}$$

- But:

$$p = \beta\gamma m_0 c \quad \Rightarrow \quad \Delta p = m_0 c\, \Delta(\beta\gamma) = m_0 c \gamma^3 \Delta\beta$$

$$\Rightarrow \frac{\Delta p}{p_0} = \gamma^2 \frac{\Delta\beta}{\beta} \quad \Rightarrow \quad \frac{\Delta L}{L_0} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0}$$

- The ballistic path length dependence on momentum is important everywhere, not just in bending magnets.
- Higher momentum particles are faster, i.e. precede the ones with lower momentum.
- The effect vanishes for relativistic particles.

Phase Slippage, Isochronicity

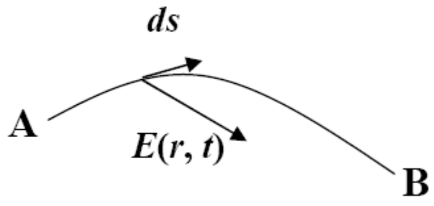
$$\frac{\Delta s}{L_0} = -\left(\frac{1}{\gamma^2} - \alpha_C\right)\frac{\Delta p}{p_0} = -\eta_C \frac{\Delta p}{p_0}$$

- Combining the previous two results we obtain the overall phase slippage factor
- If $\frac{1}{\gamma^2} = \alpha$,

the circulation time does not depend on the particle momentum any more. One calls this isochronous transport

Energy Gain/Loss

- The change in energy for a particle that moves from A to B is given by:



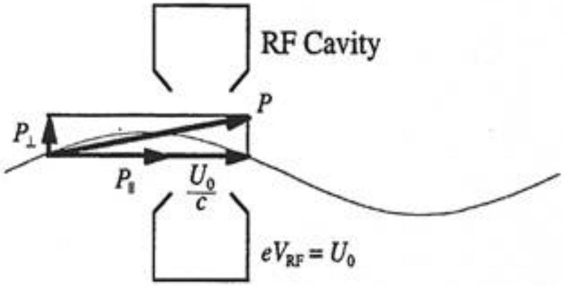
$$\Delta E = q\int_0^L \overline{E}_F\left(\overline{r},t\right)\cdot d\overline{s} = qV$$

- Now define a voltage V such that *V* depends only on the particle trajectory. It includes the contribution of every electric field (RF fields, space charge fields, fields due to the interaction with the vacuum chamber, …)
- In addition there are changes in energy *U*(*E*) that depend also on the particle energy (e.g. radiation emitted by a particle under acceleration - synchrotron radiation)
- The total change in energy is given by the sum of the two terms:

$$\Delta E_T = qV + U\left(E\right)$$

ALS example of RF cavity

- Cavities replenish the energy loss due to synchrotron radiation



Rate of Energy Change

The energy variation for the reference particle is given by:

$$\Delta E_T(s_0) = qV(s_0) + U(E_0)$$

For particle with energy $E = E_0 + \Delta E$ and orbit position $s = s_0 + \Delta s$:

$$\Delta E_T(s) = qV(s_0 + \Delta s) + U(E_0 + \Delta E) \cong qV(s_0) + q\left.\frac{dV}{ds}\right|_{s_0} \Delta s + U(E_0) + \left.\frac{dU}{dE}\right|_{E_0} \Delta E$$

Where the last expression holds for the case where $\Delta s \ll L_0$ (reference orbit length) and $\Delta E \ll E_0$.

In this approximation we can express the average rate of change of the energy respect to the reference particle energy by:

$$\frac{d\Delta E}{dt} \cong \frac{\Delta E_T(s) - \Delta E_T(s_0)}{T_0} \quad \Rightarrow \quad \frac{d\Delta E}{dt} \cong \frac{1}{T_0}\left(q\left.\frac{dV}{ds}\right|_{s_0} \Delta s + \left.\frac{dU}{dE}\right|_{E_0} \Delta E \right)$$

$$where\ \ T_0 = \frac{L_0}{\beta_0 c} \quad with \quad \begin{array}{l} L_0 = lenght\ of\ the\ reference\ orbit\ between\ A\ and\ B \\ \beta_0 c = velocity\ of\ the\ reference\ particle \end{array}$$

Synchrotron Oscillations

Define the frequency and damping terms:

$$\Omega^2 = \eta_C \frac{1}{p_0} \frac{q}{T_0} \frac{dV}{ds}\bigg|_{s_0}$$

$$\alpha_D = -\frac{1}{2T_0} \frac{dU}{dE}\bigg|_{E_0}$$

We obtain the equations of motion for the longitudinal plane:

$$\frac{d^2 \Delta s}{dt^2} + 2\alpha_D \frac{d\Delta s}{dt} + \Omega^2 \Delta s = 0$$

$$\Delta E(t) = -\frac{p_0}{\eta_C} \frac{d\Delta s}{dt}$$

$$\Delta s << L_0$$

$$\Delta E << E_0$$

Small Amplitude: Damped harmonic oscillator

$$\frac{d^2\Delta s}{dt^2} + 2\alpha_D \frac{d\Delta s}{dt} + \Omega^2 \Delta s = 0$$

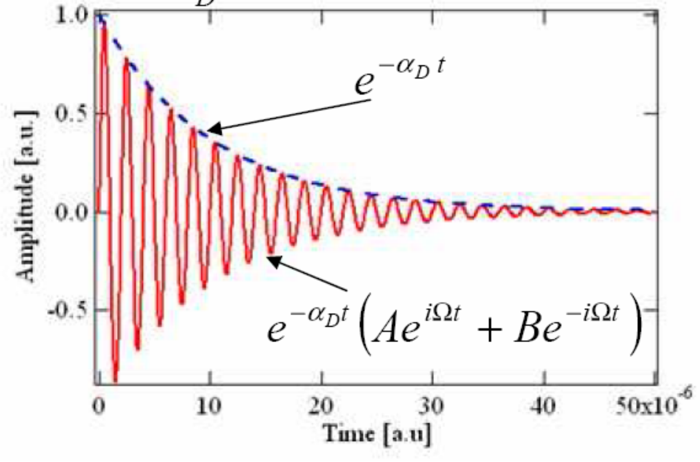
This expression is the well known damped harmonic oscillator equation, which has the general solution:

$$\Delta s(t) \cong e^{-\alpha_D t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right)$$

$$\alpha_D > 0 \quad \Leftrightarrow \quad \text{damped oscillation}$$
$$\alpha_D < 0 \quad \Leftrightarrow \quad \text{anti-damped oscillation}$$

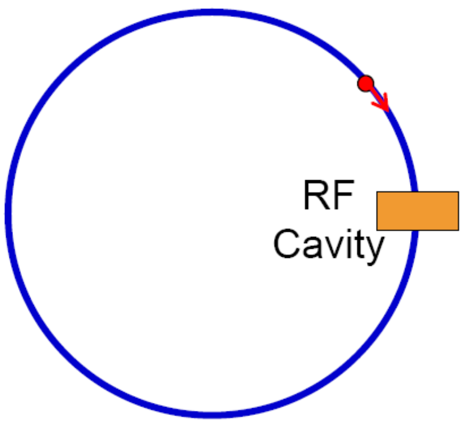
$$\Omega^2 > 0 \quad \Leftrightarrow \quad \text{stable oscillation}$$
$$\Omega^2 < 0 \quad \Leftrightarrow \quad \text{unstable motion}$$

$\alpha_D > 0 \quad and \quad \Omega^2 > 0$



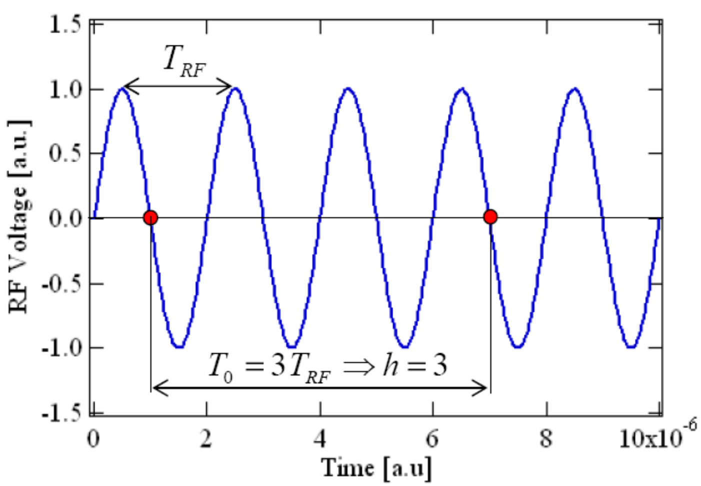
Synchronicity/Harmonic Number

- Let's consider a storage ring with reference trajectory of length L_0:



$$V_{RF}(t) = \hat{V}\sin(\omega_{RF} t)$$

$$T_0 = \frac{L_0}{\beta c} \qquad T_{RF} = \frac{1}{f_{RF}} = \frac{2\pi}{\omega_{RF}}$$



$$T_0 = hT_{RF} \Rightarrow f_0 = \frac{f_{RF}}{h}$$

Synchronicity Condition

The integer h is called the *harmonic number*

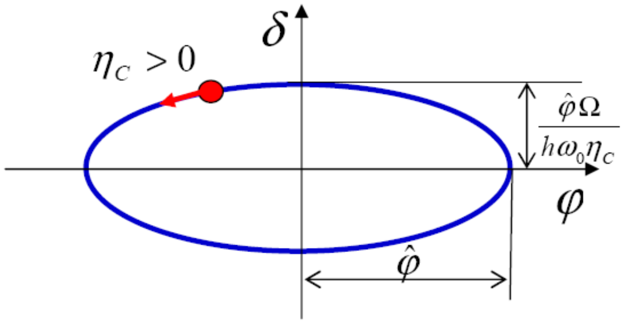
Longitudinal Phasespace

We just found:

$$\varphi = \hat{\varphi}\cos(\Omega t + \psi)$$

$$\delta = \frac{\hat{\varphi}\Omega}{h\omega_0\eta_C}\sin(\Omega t + \psi)$$

$$\frac{\varphi^2}{\hat{\varphi}^2} + \delta^2\left(\frac{h\omega_0\eta_C}{\hat{\varphi}\Omega}\right)^2 = 1$$

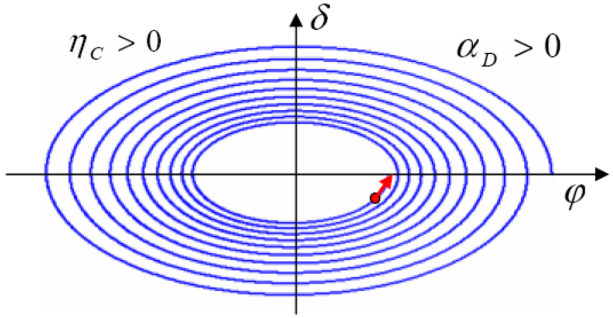


This equation represents an ellipse in the longitudinal phase space {φ, δ}

With damping:

$$\varphi = \hat{\varphi}e^{-\alpha_D t}\cos(\Omega t + \psi)$$

$$\delta = \frac{\hat{\varphi}\Omega}{h\omega_0\eta_C}e^{-\alpha_D t}\sin(\Omega t + \psi)$$



In rings with negligible synchrotron radiation (or with negligible non-Hamiltonian forces, the longitudinal emittance is conserved.

This is the case for heavy ion and for most proton machines.

Large Amplitudes/Separatrix

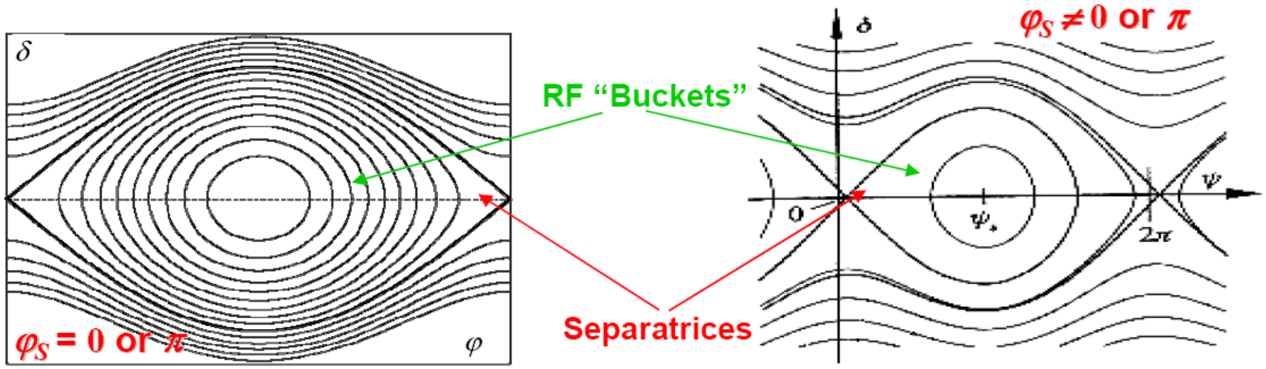
So far we have used the *small oscillation approximation* where:

$$\Delta E_T(\psi) = qV(\varphi_S + \varphi) = q\hat{V}\sin(\varphi_S + \varphi) \cong qV(\varphi_S) + q\left.\frac{dV}{d\varphi}\right|_{\varphi_S}\varphi = q\hat{V}\varphi_S + q\hat{V}\varphi$$

In the more general case of larger phase oscillations:

$$\Delta E_T(\psi) = qV(\varphi_S + \varphi) \cong q\hat{V}\sin(\varphi_S + \varphi)$$

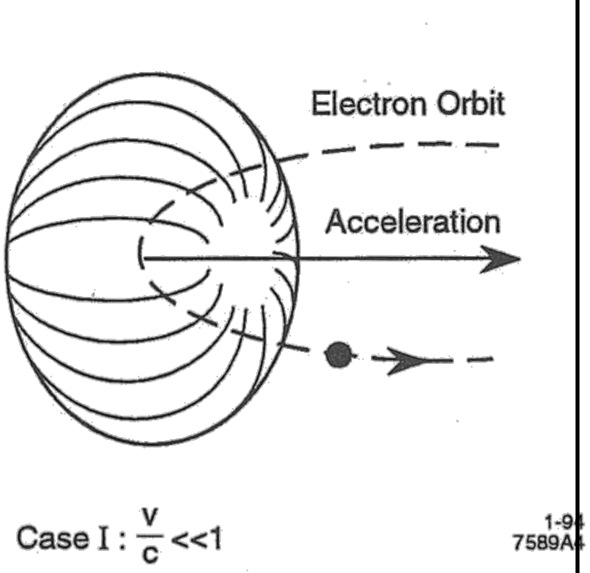
And by Numerical integration:



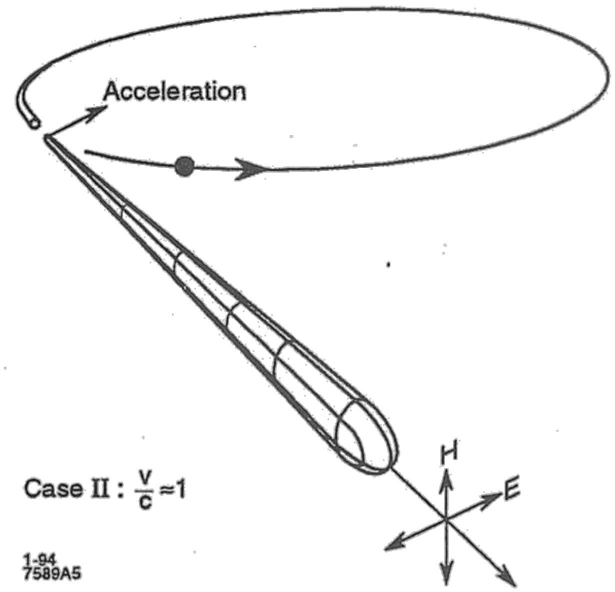
- **For larger amplitudes, trajectories in the phase space are not ellipsis anymore.**
- **Stable and unstable orbits exist. The two regions are separated by a special trajectory called *separatrix***
- **Larger amplitude orbits have smaller synchrotron frequencies**

Synchrotron Radiation

- Radiated power increases at higher velocities
- Radiation becomes more focused at higher velocities



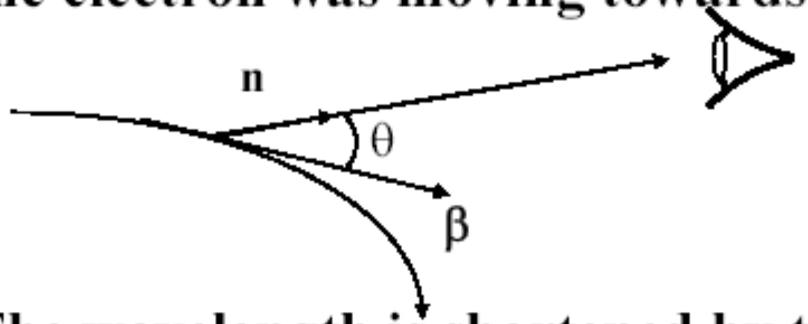
At low electron velocity (non-relativistic case) the radiation is emitted in a non-directional pattern



When the electron velocity approaches the velocity of light, the emission pattern is folded sharply forward. Also the radiated power goes up dramatically

Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) T_{emit}$$

The wavelength is shortened by the same factor

$$\lambda_{obs} = (1 - \beta \cos\theta) \lambda_{emit}$$

in ultra-relativistic case, looking along a tangent to the trajectory

$$\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}$$

since

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \cong \frac{1}{2\gamma^2}$$

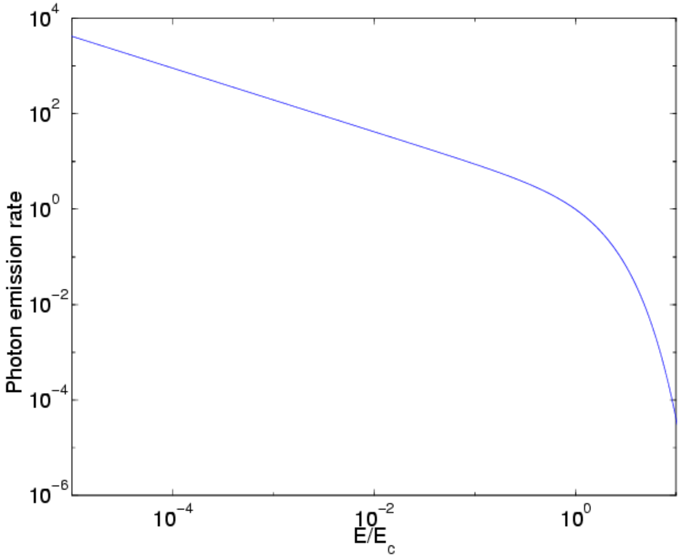
Radiation

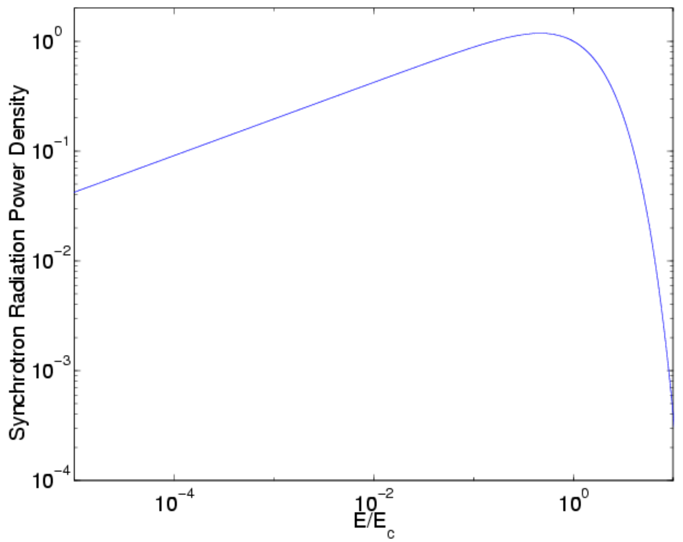
The power emitted by a particle is

$$P_{SR} = \frac{2}{3}\alpha \hbar c^2 \frac{\gamma^4}{\rho^2}$$

and the energy loss in one turn is

$$U_0 = \frac{4\pi}{3}\alpha \hbar c \frac{\gamma^4}{\rho}$$





Radiation damping

Energy damping:

$$\alpha_D > 0 \qquad \alpha_D = -\frac{1}{2T_0}\frac{dU}{dE}\bigg|_{E_0} \quad \Rightarrow \quad \frac{dU}{dE}\bigg|_{E_0} < 0$$

Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha \hbar c^2 \frac{\gamma^4}{\rho^2}$$

- **Typically, synchrotron radiation damping is very efficient in electron storage rings and negligible in proton machines.**
- **The damping time $1/\alpha_D$ (~ ms for e-, ~ 13 hours LHC at 7 TeV) is usually much larger than the period of the longitudinal oscillations $1/2\pi\Omega$ (~ μs). This implies that the damping term can be neglected when calculating the particle motion for $t \ll 1/\alpha_D$:**

$$\frac{d^2\Delta s}{dt^2} + \Omega^2 \Delta s = 0$$

Harmonic oscillator equation

Radiation damping

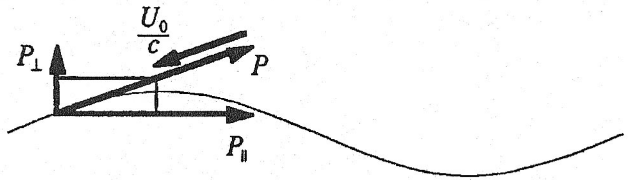
Energy damping:

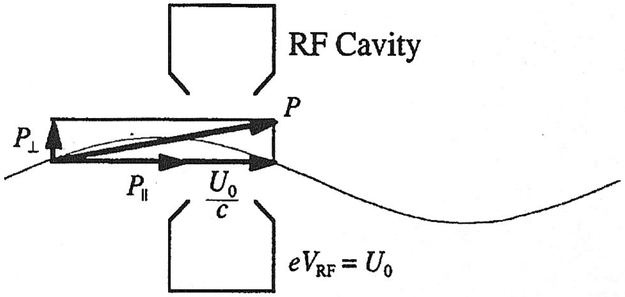
Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3}\alpha \hbar c^2 \frac{\gamma^4}{\rho^2}$$

Transverse damping:

Energy loss is in the direction of motion while the restoration in the s direction





Quantum excitation - Longitudinally

The synchrotron radiation emitted as photons, the typical photon energy is

$$u_c = \hbar\omega_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho}$$

The number of photons emitted is

$$N = \frac{4}{9}\alpha c\frac{\gamma}{\rho}$$

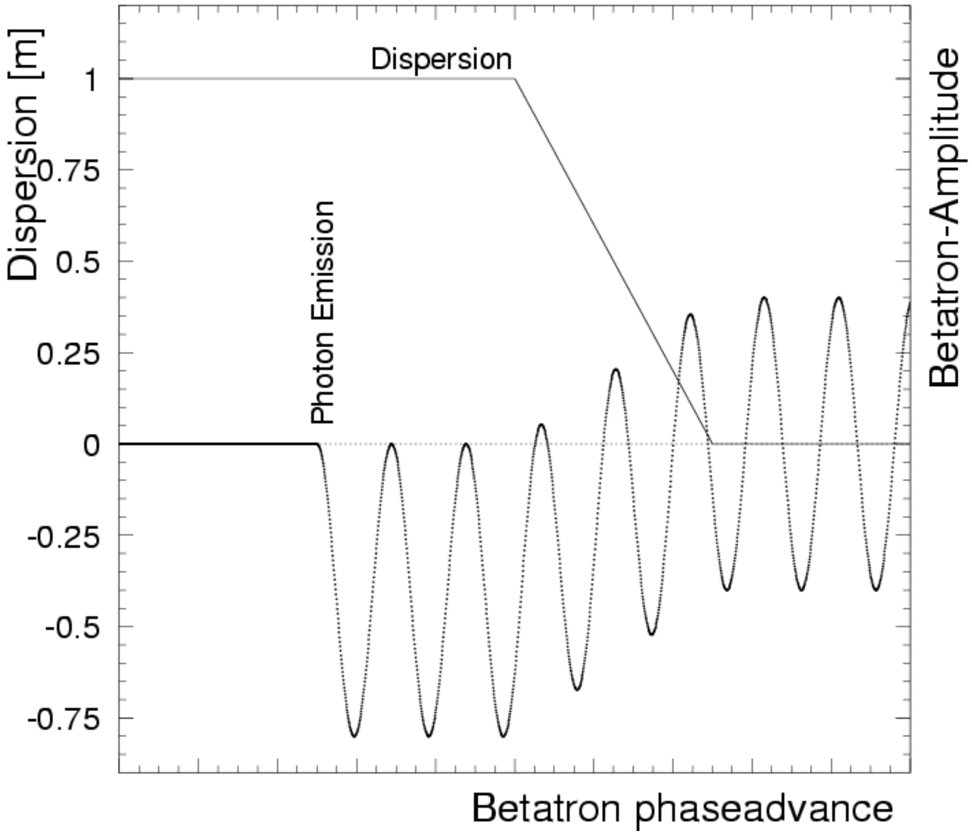
With a statistical uncertainty of $\sqrt{N}$

The equilibrium energy spread and bunch length is

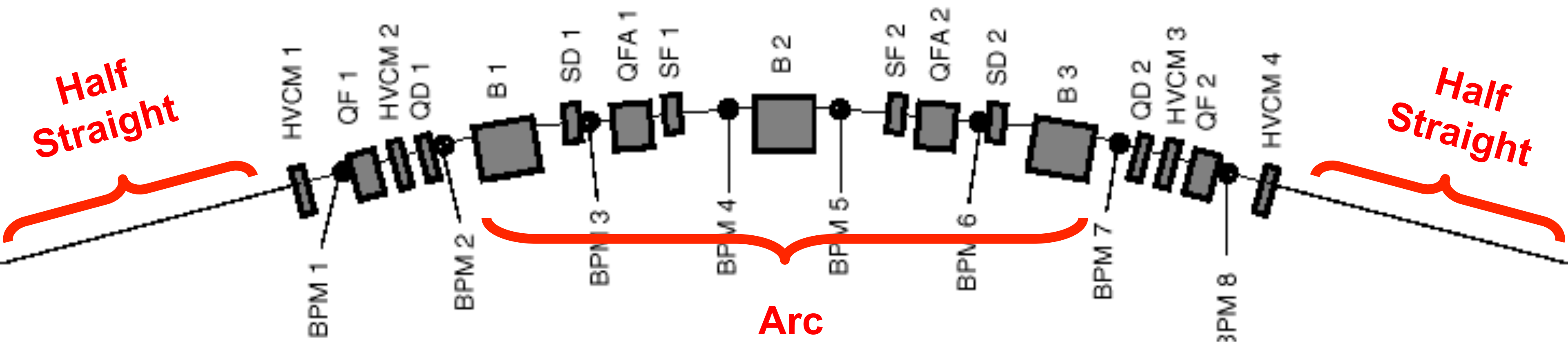
$$\left(\frac{\sigma_e}{E}\right)^2 = 1.468\cdot 10^{-6}\frac{E^2}{J_\varepsilon \rho} \text{ and } \sigma_L = \frac{\alpha R}{f_0}\sigma_e$$

Quantum Excitation - Transversely

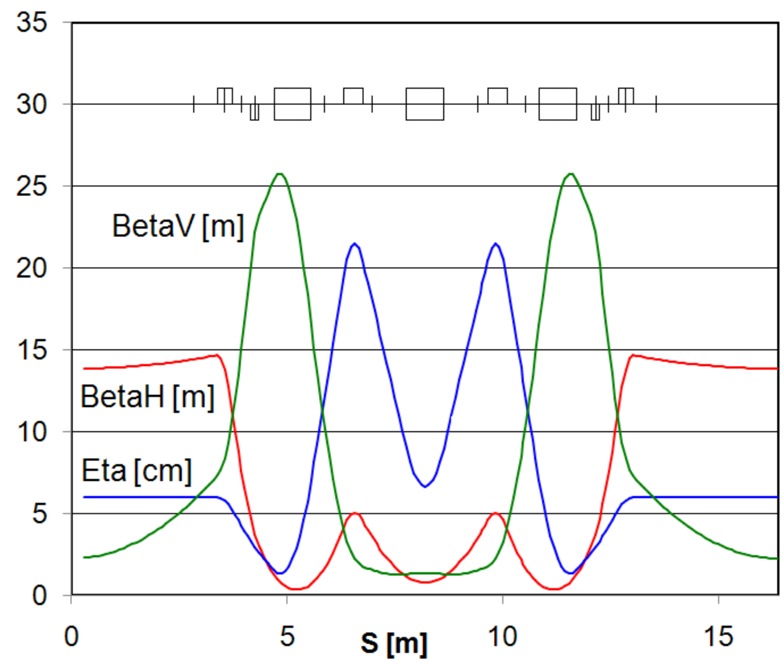
Particles change their energy in a region of dispersion undergoes increase transverse oscillations. This balanced by damping gives the equilibrium emittances.



Low Emittance Lattice Example: ALS



- The ALS is an example of a low emittance lattice
- It is a so called triple bend achromat and minimizes the dispersion at the location where synchrotron radiation is emitted (dipoles)
- Those achromat lattices were the major advance in 3rd generation light sources



$$\text{Emittance} \propto \int_{Bend} \frac{\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2}{\rho^3} ds$$

Time Scales for Particle Dynamics in Rings

- At this point we have discussed the motion of a particle in an accelerator for all 6 phase space dimensions (4 transverse dimensions and 2 longitudinal ones)
- An important effect is that the time scales for different phenomena are quite different:
  - Damping: several ms for electrons, ~ infinity for heavier particles
  - Betatron oscillations: ~ tens of ns
  - Synchrotron oscillations: ~ tens of μs
  - Revolution period: ~ hundreds of ns to μs

Summary

- Recapped many concepts of linear beam dynamics
- Matrix (transverse) beam transport approach is helpful to calculate simple problems by hand
- Computer codes use an extension of this approach (nonlinear integrators for individual elements, symplectic, …)
- Find closed orbit – generate map around closed orbit – lattice functions
- Historic and text book approach of hill's equation is not very useful for practical purposes (either calculations by hand or computer codes)

Thanks to David Robin and Fernando Sannibale for some material used in this lecture

List of Literature/Text Books

- Particle Accelerator Physics I (2nd edition, 1998), by Helmut Wiedemann, Springer (part II: nonlinear and … is beyond the scope of this lecture)
- D.A. Edwards and M.J. Syphers, An Introduction to the Physics of High Energy Accelerators, John Wiley & Sons (1993)
- Accelerator Physics, S.Y. Lee, World Scientific, Singapore, 1999 (ISBN 9810237103)
- Many nice proceedings of CERN accelerator schools can be found at http://cas.web.cern.ch/cas/CAS_Proceedings.html , for the purpose of this class especially CERN 94-01 v1 + v2
- Material for further reading (if you got really interested):
  - Particle Accelerator Physics II, H. Wiedemann, Springer (nonlinear …)
  - CERN 95-06 v1 + v2 (Advanced Class)
  - CERN 98-04 (Synchrotron Radiation+Free Electron Lasers)
  - Physics of Collective Beam Instabilities …, A.W. Chao, John Wiley and Sons, New York, 1993 (ISBN 0471551848) and http://www.slac.stanford.edu/~achao/wileybook.html