USPAS 2012: Grand Rapids, MSU

Nonlinear Lattice Characterization:
Dynamic Aperture, Momentum Aperture, Lifetime, Frequency Map Analysis

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Outline

How (transverse) single particle dynamics determines injection efficiency, lifetime, …

– Motivation

– Nonlinear Dynamics – Dynamic Aperture

– Tunescans

– Frequency Maps

  • On energy dynamic aperture: Injection Efficiency

– Lifetime limiting processes

  • Momentum aperture: Touschek Lifetime

– Summary

Introduction

• *Beam often needs to be stored for as long as possible (stability, flux, luminosity), making lifetime a key performance parameter*

• Particles are lost in accelerators because of finite apertures. Limiting aperture can be *physical* or *dynamic*:

  • Vacuum chamber → physical aperture

  • Nonlinear single particle dynamics → dynamic (energy) aperture

• Important processes include: *elastic and inelastic gas scattering, intra beam scattering, quantum lifetime (SR), tune resonances, etc.*

  •They can increase particle oscillation amplitudes (e.g. scattering, diffusion) ultimately leading to particle loss on physical apertures.

  • Damping and excitation plays a major role in the electron/positron case.

• Maximizing lifetime has important consequences for machine design, e.g. minimizing residual gas scattering requires ultra high vacuum technologies.

Motion at large Amplitudes/Dynamic Aperture

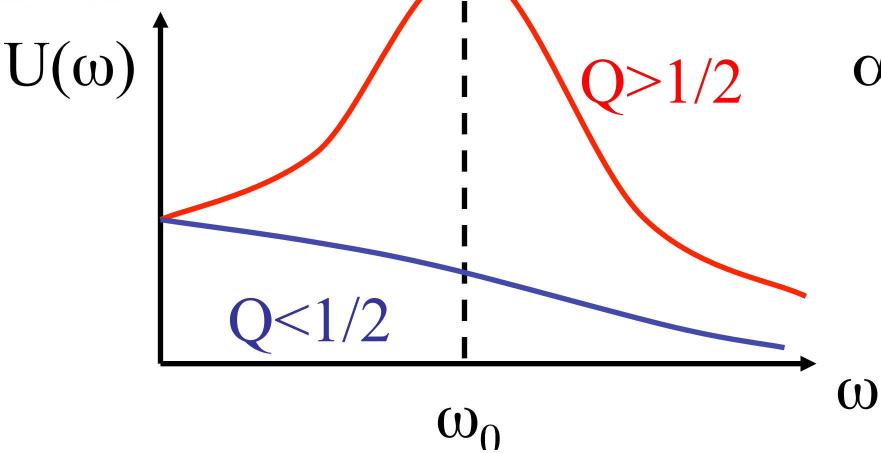
In many cases the lifetime (and injection efficiency) can not be described by physical apertures or RF-bucket height only!

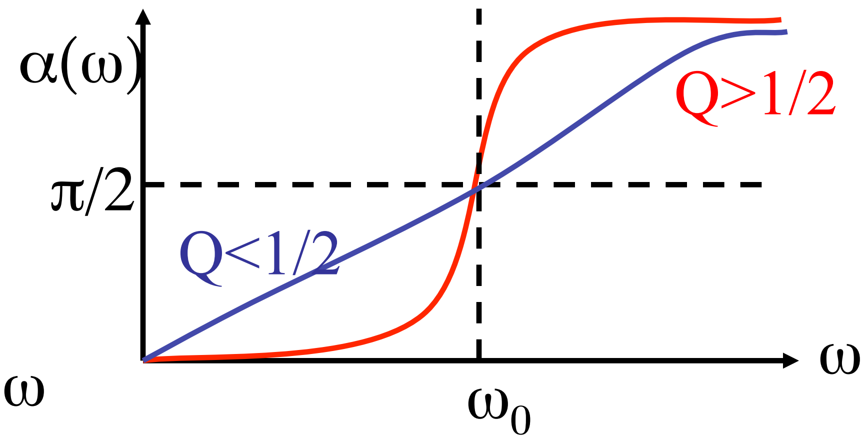
The stability of the motion of particles at large amplitudes is also important. It often determines the performance of a storage ring.

- Dynamic Aperture
  - Injection Efficiency
  - Lifetime
- Dynamic Momentum Aperture
  - Lifetime

Therefore we need to understand the nonlinear beam dynamics

Resonance effect



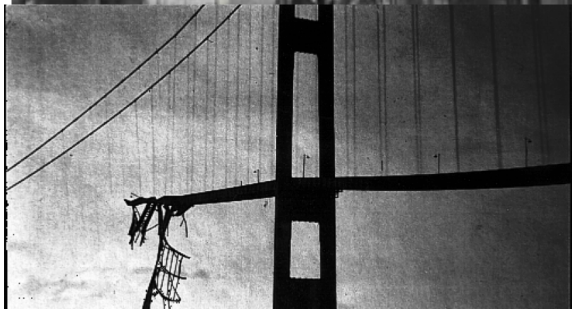


$$U(\omega) = \frac{U(0)}{\sqrt{(1-(\frac{\omega}{\omega_0})^2)^2 + (\frac{\omega}{Q\omega_0})^2}}$$

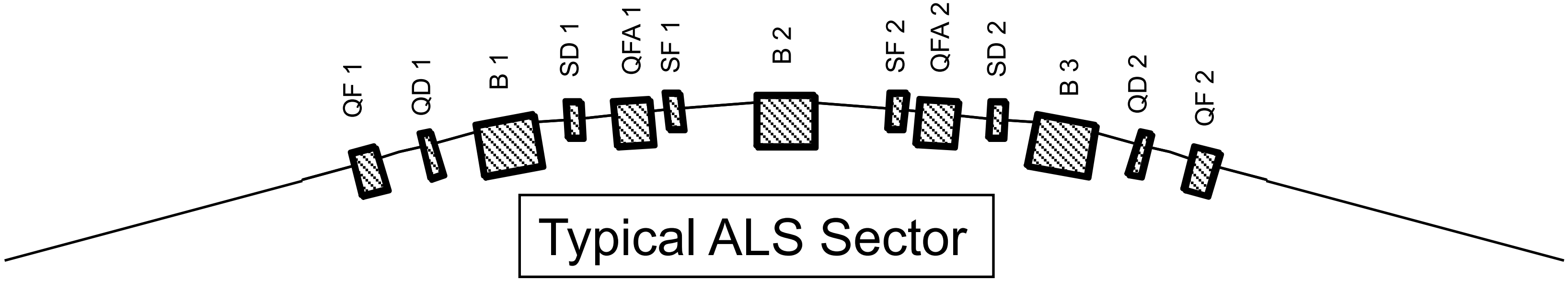
- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

strong wind excited motion at the Eigenfrequencies



Advanced Light Source



- ALS consists of 12 sectors
  - 12-fold periodicity $\Longrightarrow$ *Suppression of resonances*

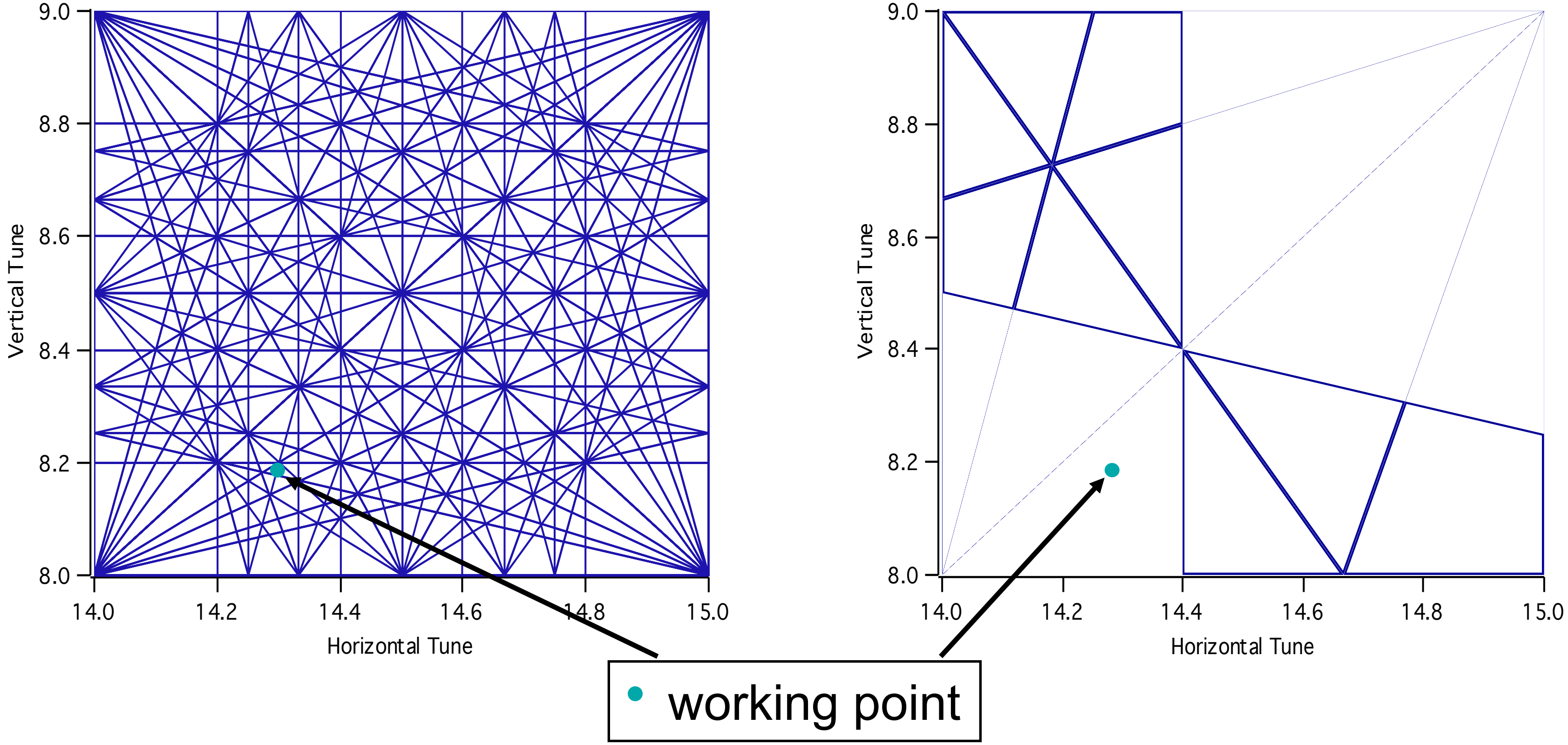
$$m\nu_x + n\nu_y = \mathbf{12} \times q$$

where m, n and q are integers

Benefits of Periodicity

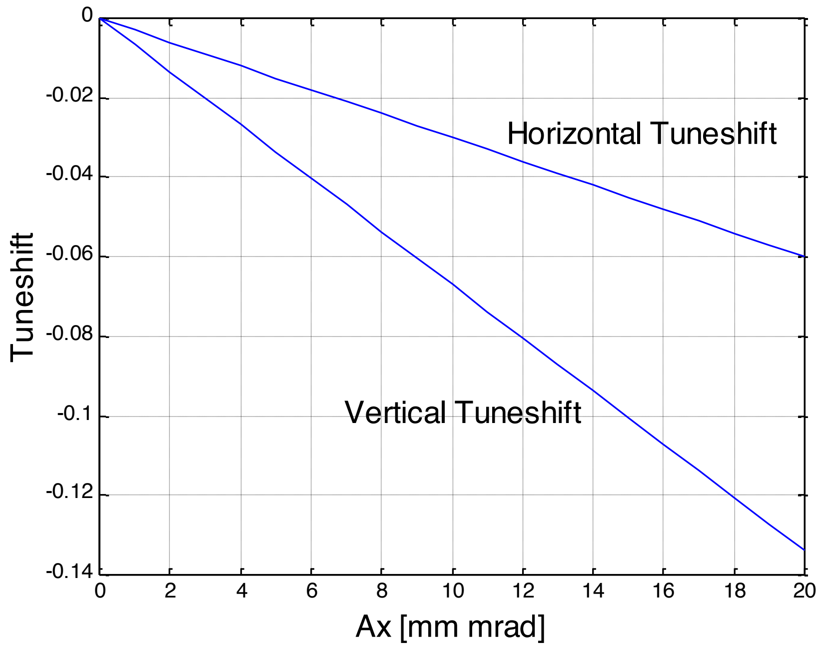
Tune plane - *resonances up to 5th order*

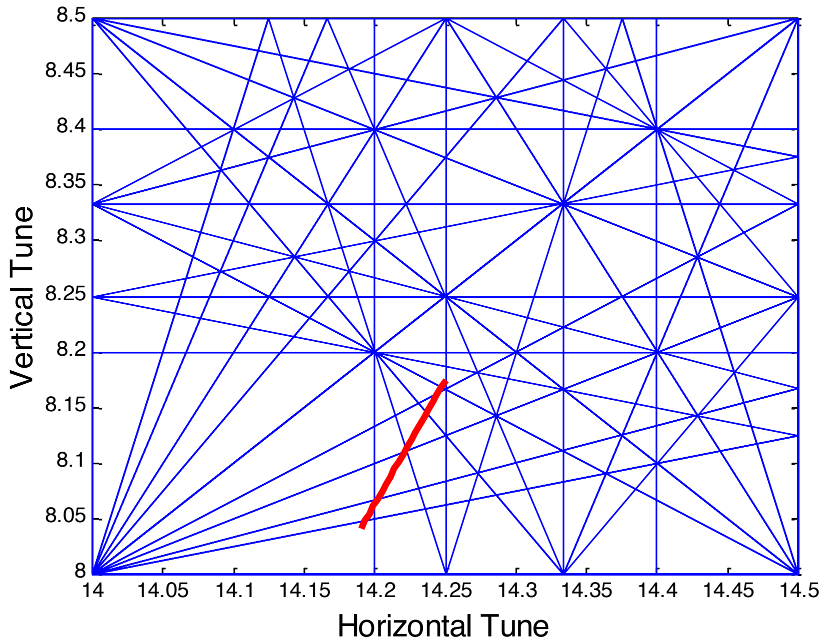
All resonances   Allowed resonances



Tune shift with amplitude

Particle tune get shifted with amplitude





Resonance Excitation

Resonances can lead to irregular and chaotic behavior for the orbits of particles which eventually will get lost by diffusion in the outer parts of the beam.

Rule of thumb => Avoid low order resonances **(<~ 12th for protons and <~ 4th for electrons)**

Unfortunately there is no simple way to forecast the real strength of a resonances without using a tracking code or through measurements

=> Tune scans
=> Frequency Map Analysis

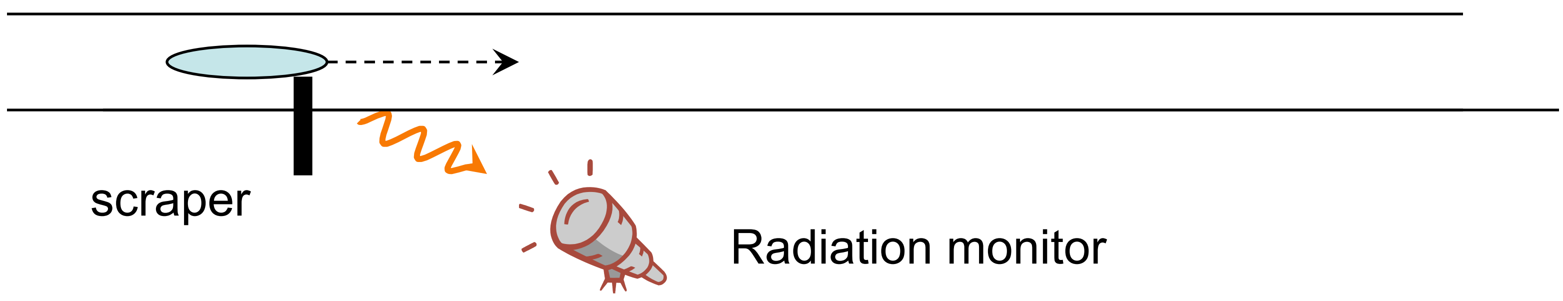
Tune scan

When resonances are present they may change the distribution of the beam at large amplitudes.

- In the case of a resonance island →particles may get trapped at large amplitudes

Technique:

- By Introducing a scraper and a loss monitor

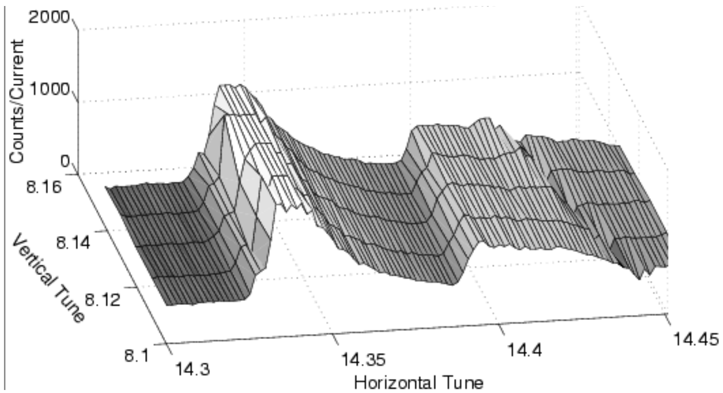


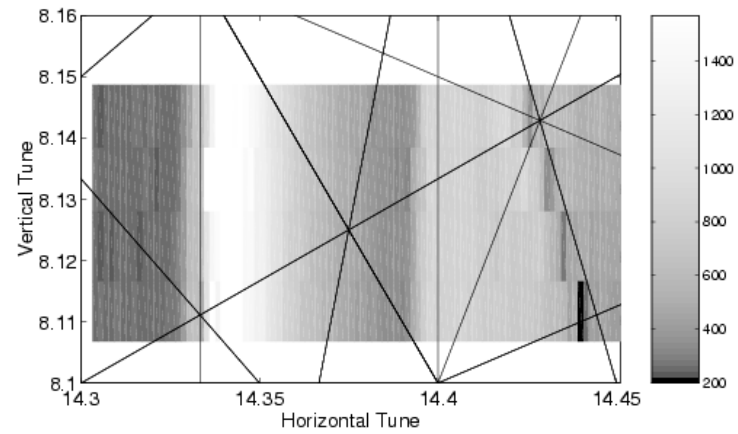
- Scan the tunes and measure the change in the count rate

Developed by A. Temnykh (Proc. Of the IXth ALL-Union Meeting on Accelerators of Chaged Particles, Dubna, 1984, INP Peport No. INP 84-131

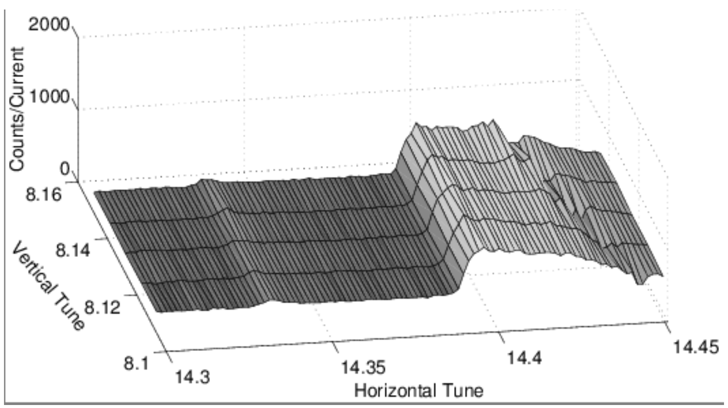
Tune scans (with and without large beta beating)

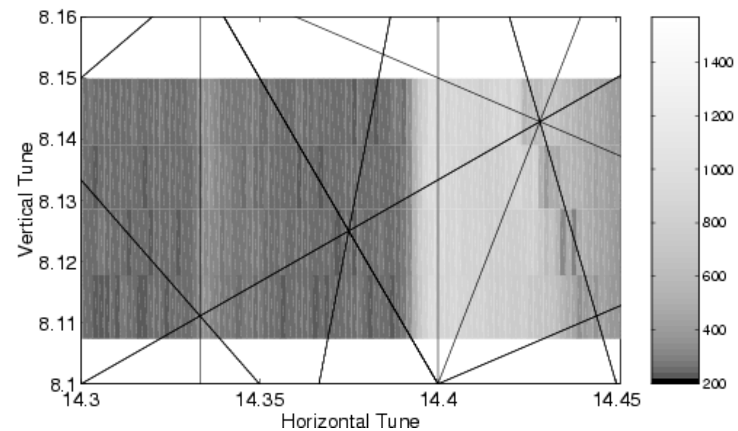
Uncorrected lattice





Corrected lattice





Three resonances are present:

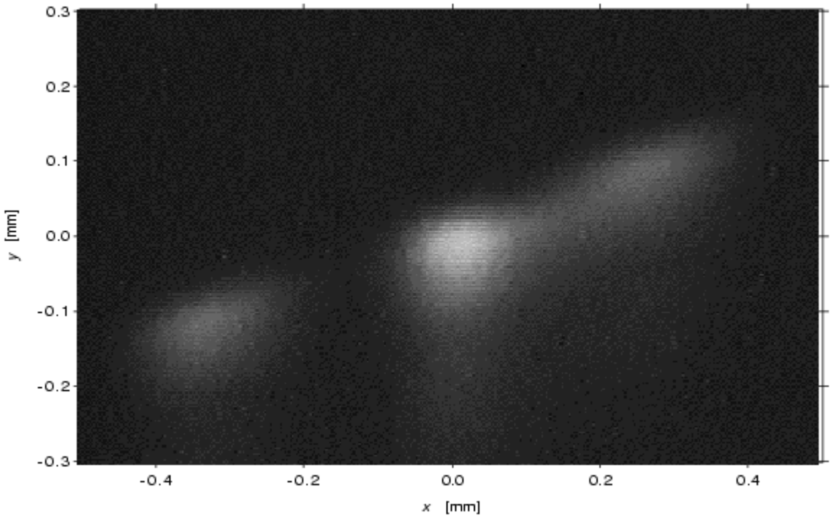
$5\nu_x = 72$ **(allowed)**

$3\nu_x = 43$ **(unallowed)**

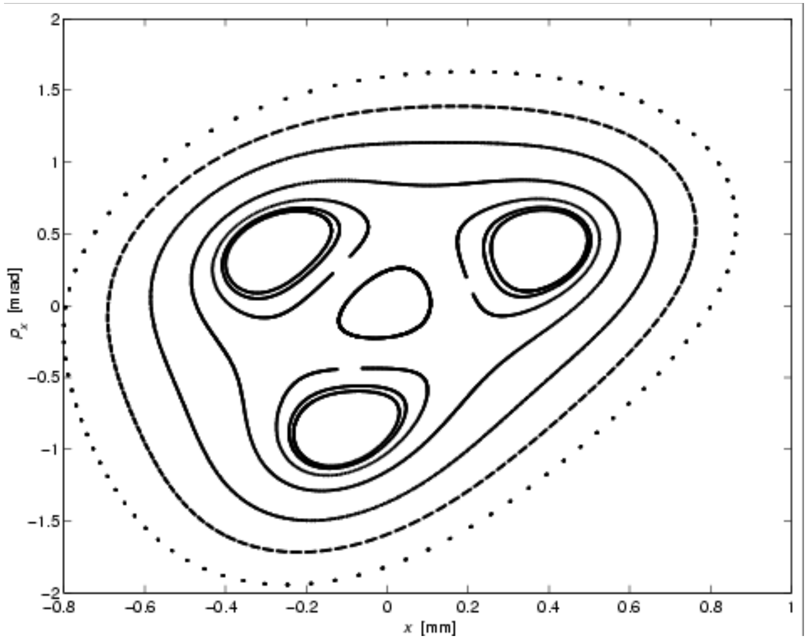
$2\nu_x - \nu_y = 37$ **(unallowed)**

Profile measurement near 3rd order resonance

Profile measurement



Horizontal phase space



Tune scan summary

Advantages

Quickly and sensitively see excited resonances in the tails and core of the beam as a function of different tunes

Disadvantages

Probing different machines and not looking at the effect of resonances on one working point and at different amplitudes. This is what one really would like to see.

Other Tools and Techniques

Other tools and techniques for studying the dynamics used at the ALS

- **Response matrix analysis (LOCO) to calibrate the linear model**
- **Symplectic integration and Frequency Map Analysis**
  - **Simulate the nonlinear dynamics and to get a global view of the dynamics (J. Laskar)**
- **Single turn kickers and BPMs, DCCT and RF scans**
  - **Test the model predictions**
  - **Model independent determination of the dynamics**

KAM Theorem

(the basis of frequency map analysis)

According to the KAM theorem, in the phase space that is sufficiently close to an integrable conservative system, many invariant tori will persist. Trajectories starting on one of these tori remain on it thereafter, executing **quasiperiodic motion with a with a fixed frequency vector** depending only on the torus.

Frequency Map Analysis

Developed by Jacques Laskar

The frequency analysis algorithm (NAFF) is a postprocesser for particle tracking data that numerically computes, over a finite time span, a frequency vector for any initial condition.

Frequency Map: Initial condition → Frequency vector

Based on the KAM theorem, frequency map analysis determines whether an orbit is regular or chaotically diffusing.

Regular orbits → Frequency vector remains fixed in time

Nonregular orbits → Frequency vector changes in time

Tunes and Diffusion Rates

TRACKING CODE + FREQUENCY ANALYSIS POSTPROCESSOR

Track particle for **N** turns

↓

Compute horizontal and vertical tunes ν_{x1} and ν_{y1}

↓

Track particle for another **N** turns

↓

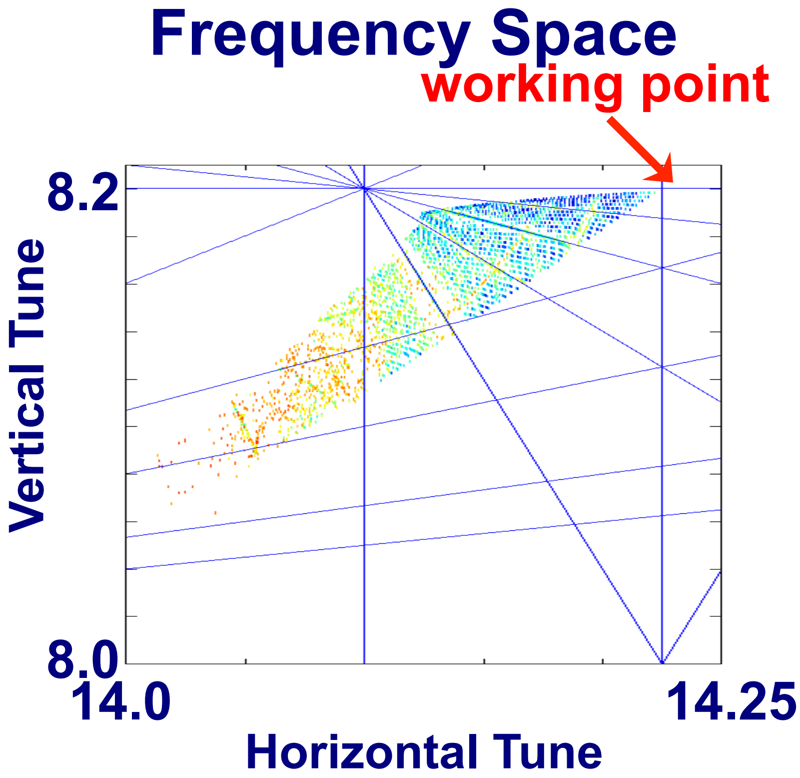
Compute horizontal and vertical tunes ν_{x2} and ν_{y2}

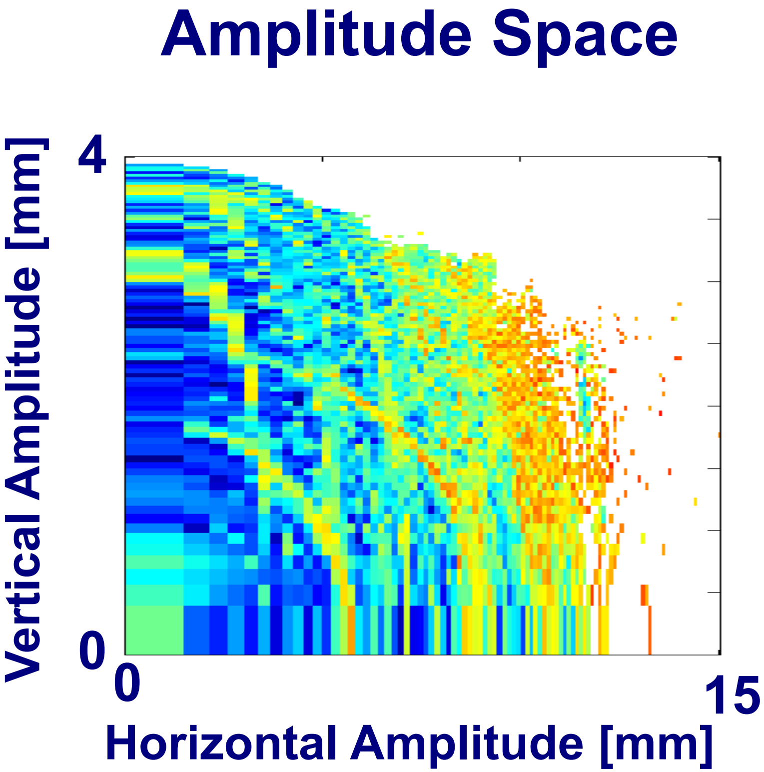
→ Compute diffusion rates

$$\frac{\partial \nu_x}{\partial \tau} \approx \frac{\nu_{x2} - \nu_{x1}}{N}$$

$$\frac{\partial \nu_y}{\partial \tau} \approx \frac{\nu_{y2} - \nu_{y1}}{N}$$

Frequency Map Analysis





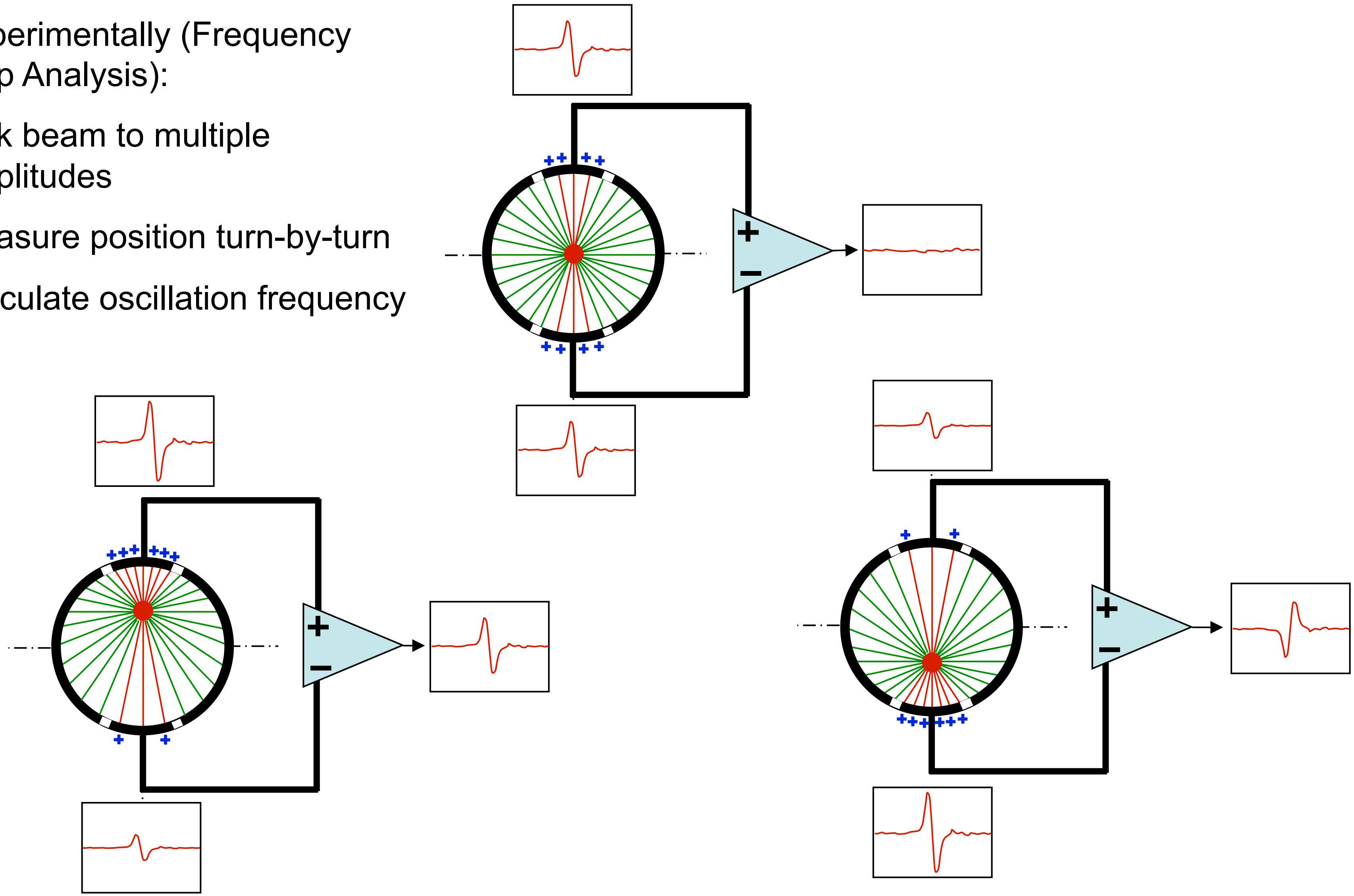
Electromagnetic Beam Position Monitors

Experimentally (Frequency Map Analysis):

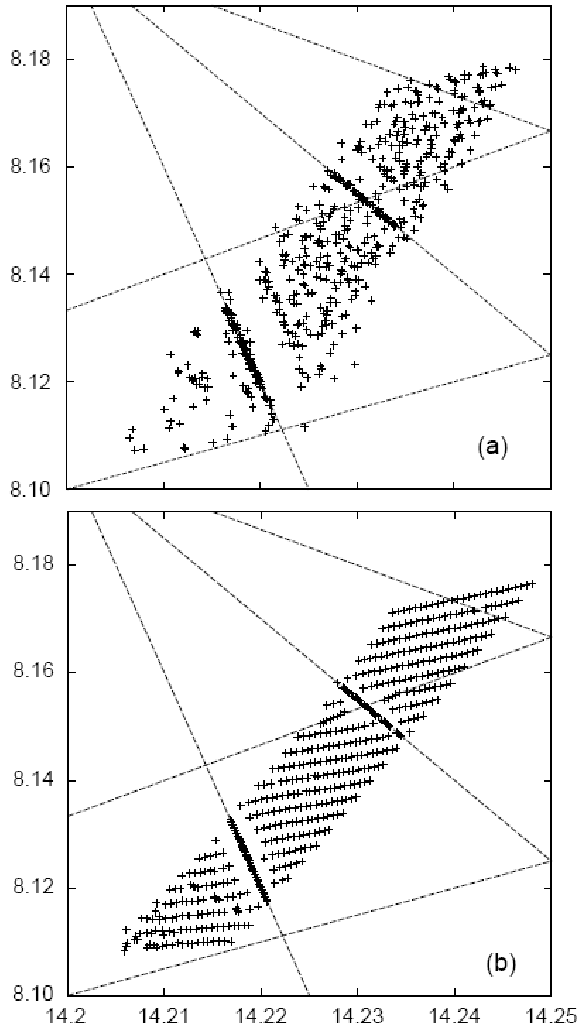
Kick beam to multiple amplitudes

Measure position turn-by-turn

Calculate oscillation frequency



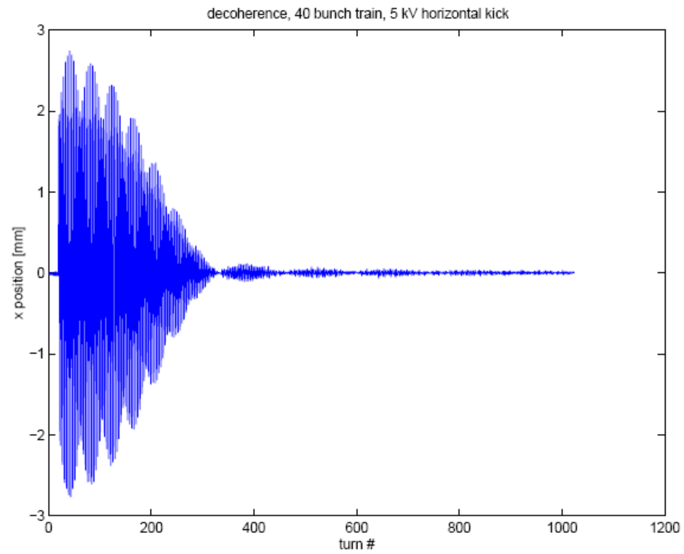
Measured Frequency Map

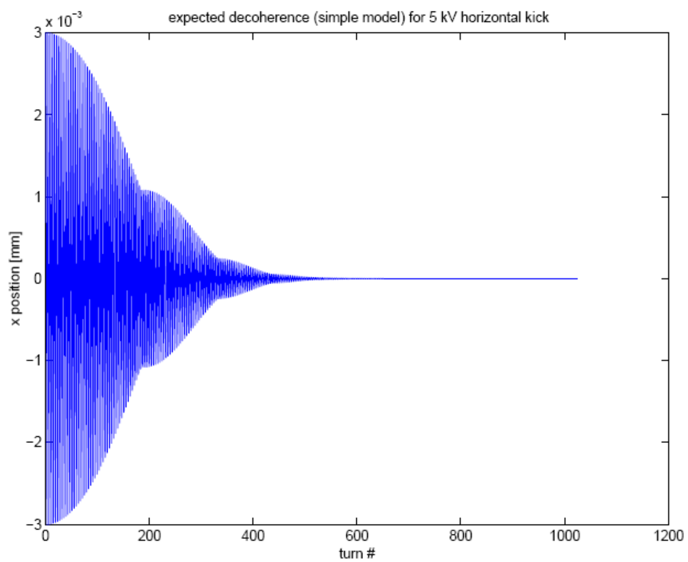


- excellent agreement, using calibrated model (gradient errors), random skew errors, nominal sextupoles

Phys. Rev. Lett. 85, 3, (July 2000), pp.558-561

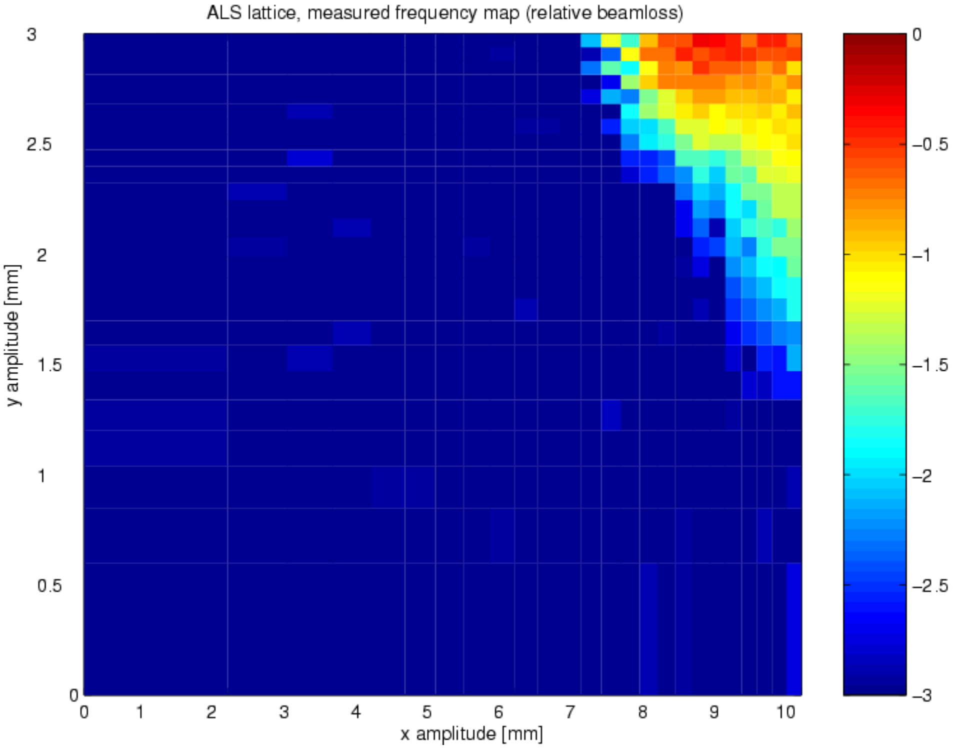
Fast Decoherence Problem for Experiment

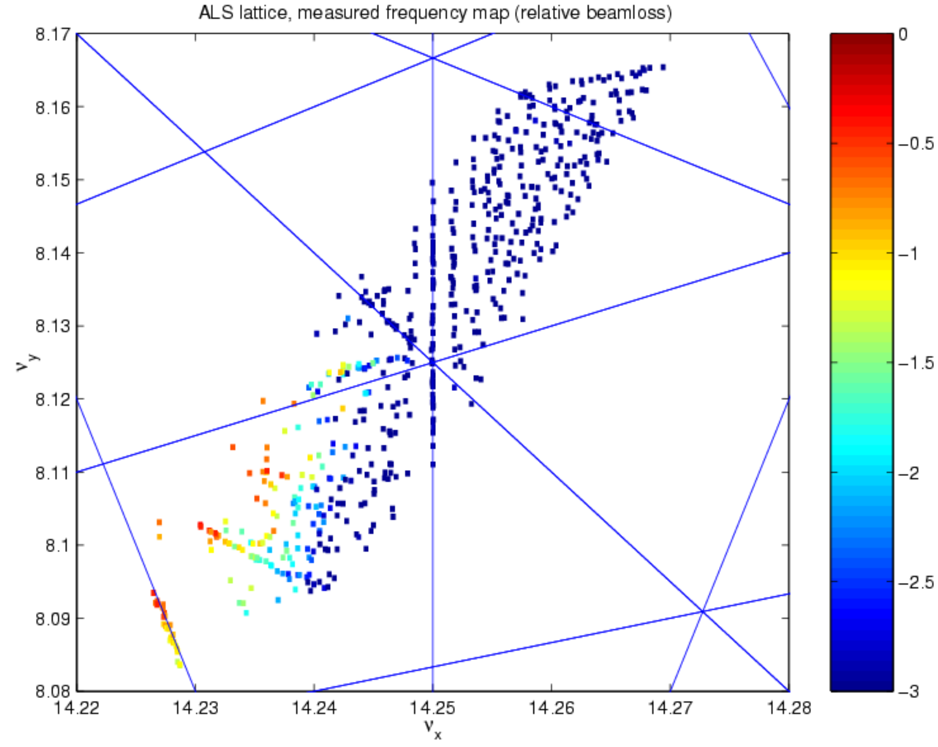




- Detuning with amplitude causes very fast decoherence for larger amplitudes
- Individual particles are still oscillating with same amplitude (radiation damping time >10k turns)
- Makes frequency analysis difficult
  - Small number of turns
  - Signal not quasiperiodic

Measured Frequency Map/Beam Loss





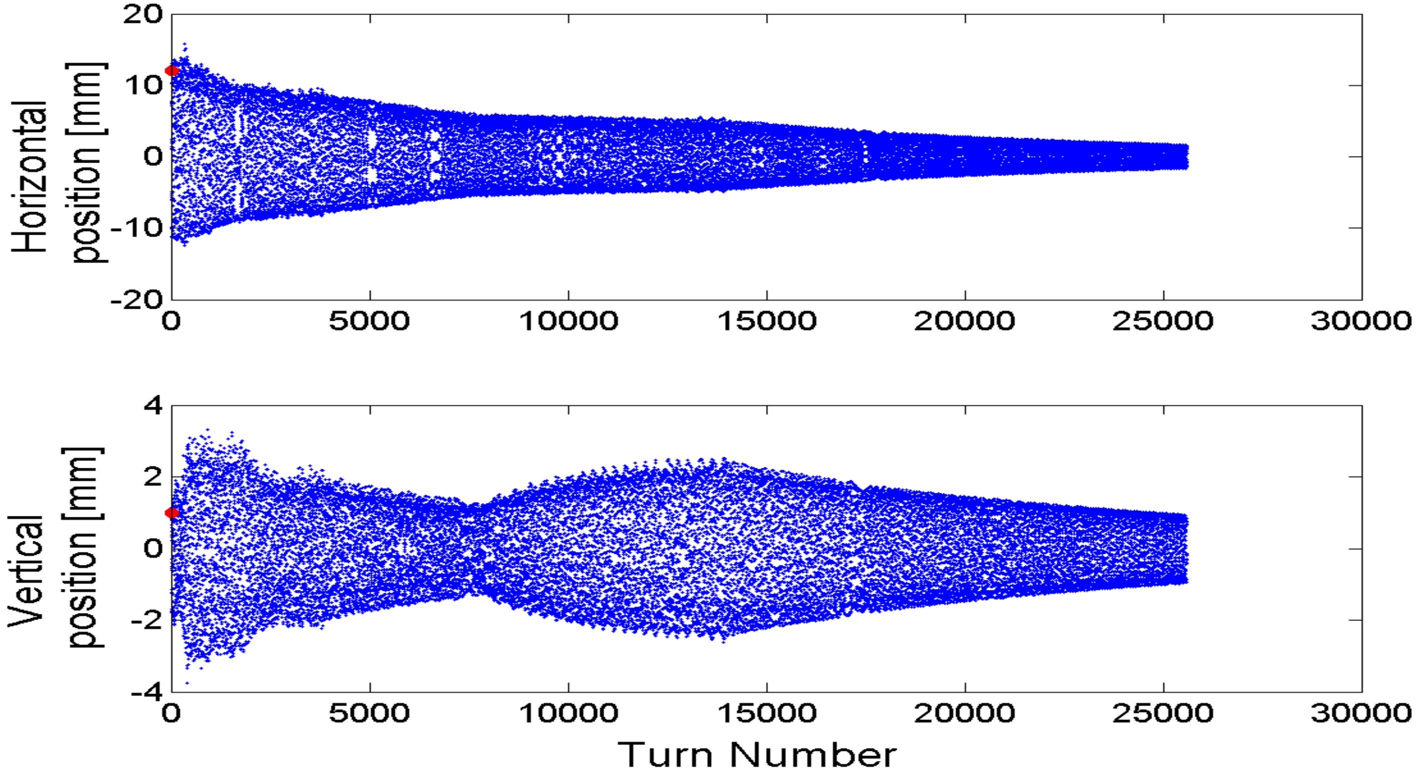
- **Partial Beam Loss mostly if particles have to pass (radiation damping) through resonance intersection**
- **Isolated resonances not dangerous.**

Side remark: Spectra contain more information than just fundamental frequencies – other resonance lines – resonance strength versus amplitude (see R. Bartolini, et al.).

Vertical orbit diffusion – On-energy example

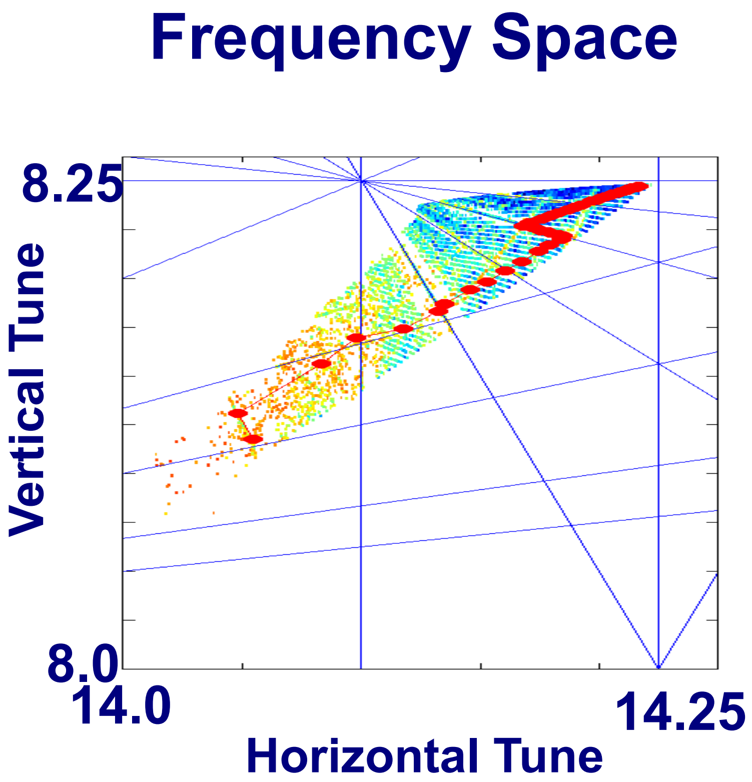
Particle are lost in the vertical plane

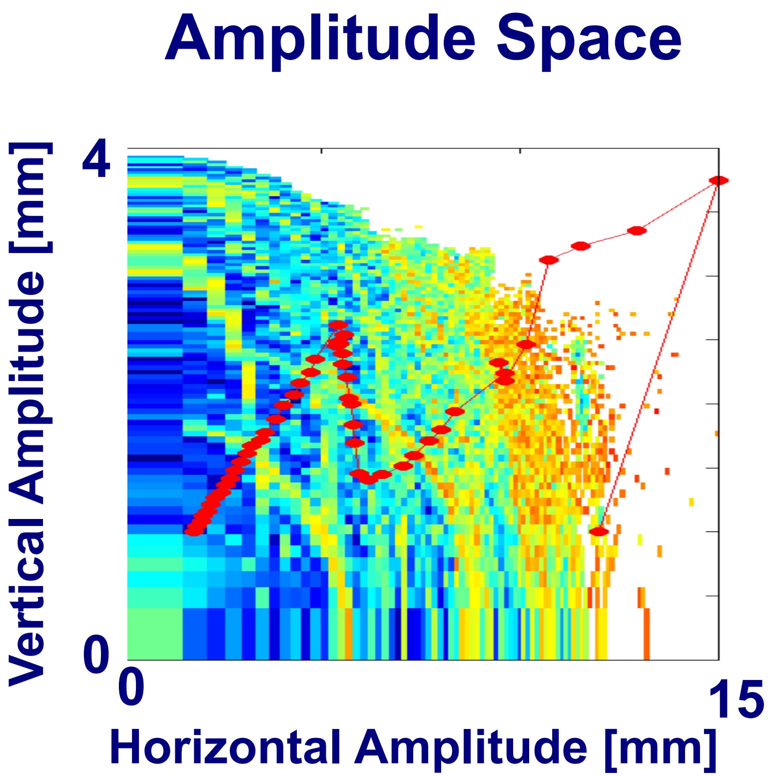
- ***via nonlinear coupling and diffusion of the trajectory.***



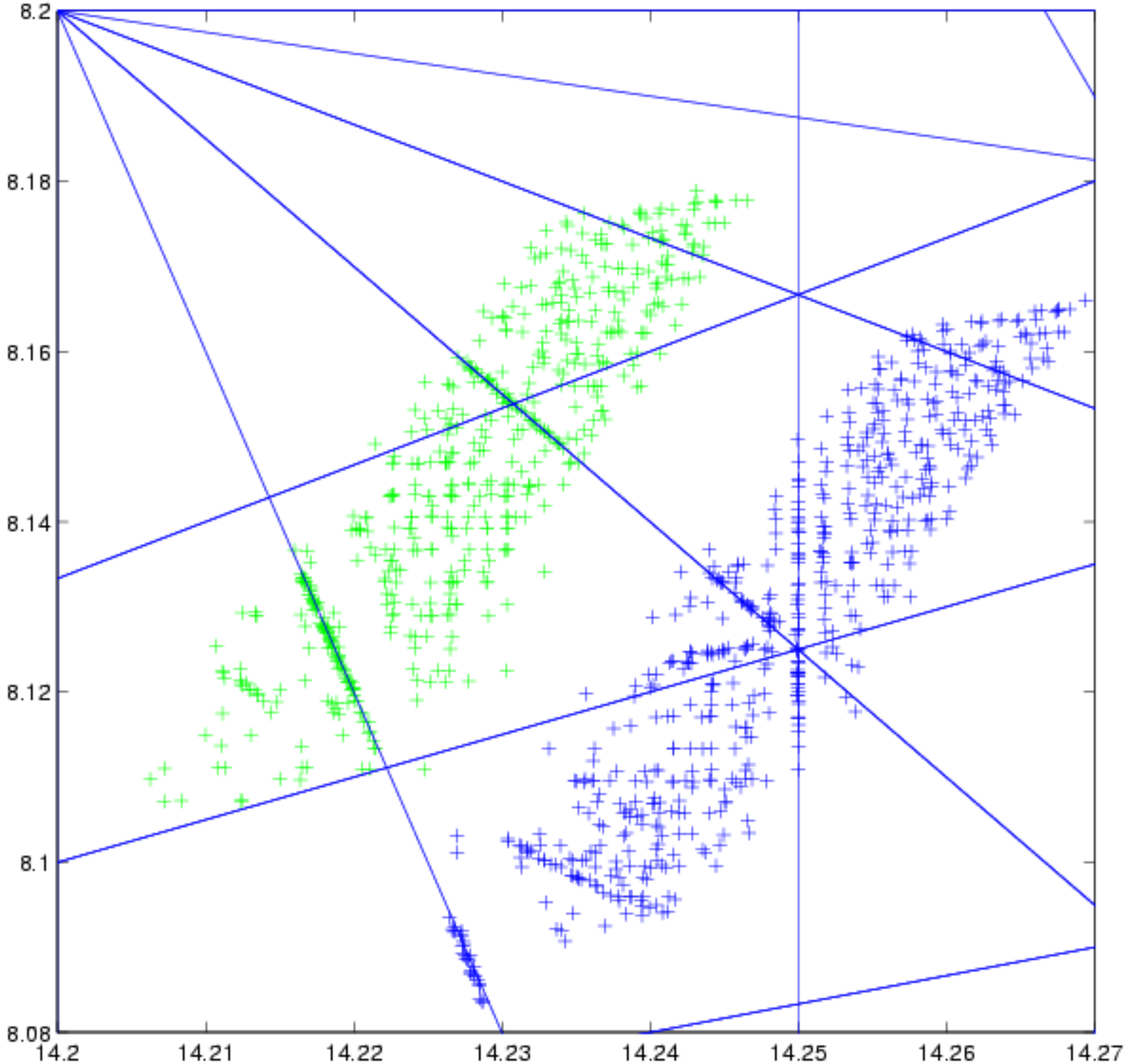
Example : Particle launched at 12 mm horizontally and 1 mm vertically and tracked with damping and synchrotron oscillations. (Simulated injection)

Frequency Map Analysis



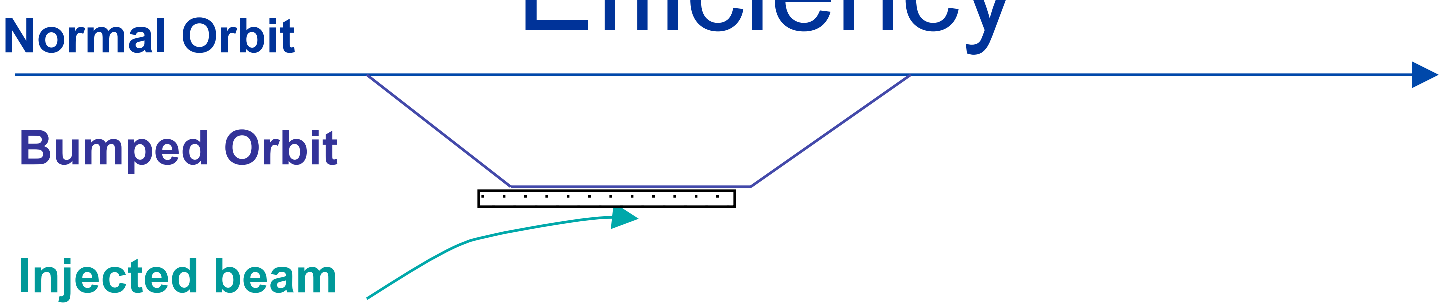


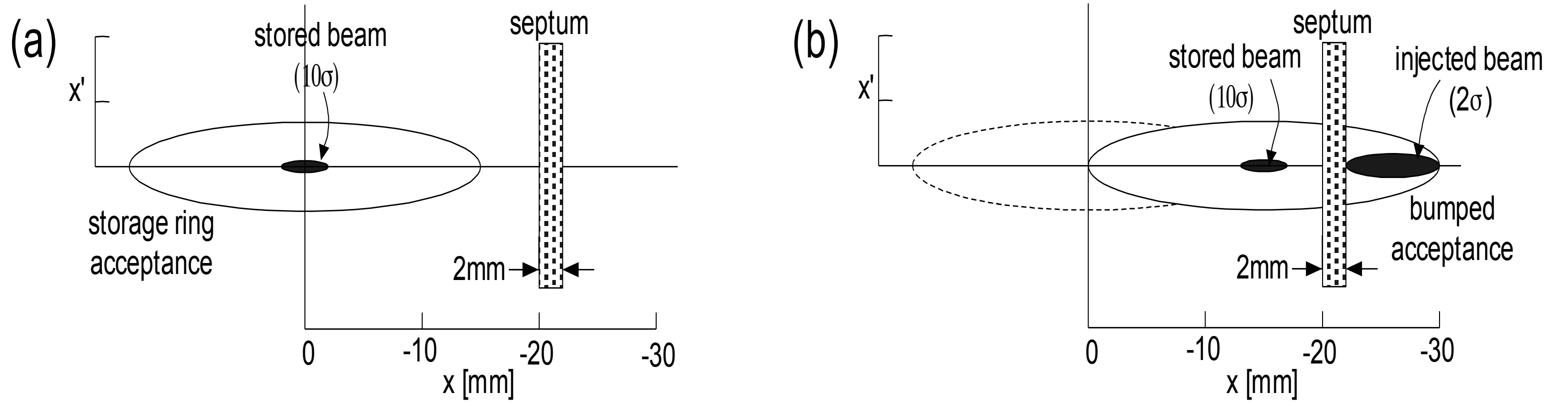
Model independent evaluation of dynamics



- Frequency map analysis allows to model indepently evaluate how regular beam motion is

Related Problem: Injection Efficiency





Storage ring physical and dynamic aperture has to be large enough to capture sufficient amount of injected beam – Often limited by dynamic aperture.

Definition of Lifetime

- In a loss process, the number of particles lost at the time *t* is proportional to the number of particles present in the beam at the time *t*:

$$dN = -\alpha N(t)dt \qquad with \quad \alpha \equiv constant$$

- By defining the lifetime τ as:

$$\tau = \frac{1}{\alpha} \quad \Rightarrow \quad N = N_0 e^{-t/\tau}$$

- From the last equation, one can see that the lifetime is defined as the time required for the beam to reduce its number of particles to 1/*e* of the initial value.
- Lifetime due to the individual effects (gas, Touschek, …) can be similarly defined. The total lifetime will be then obtained by summing the individual contributions:

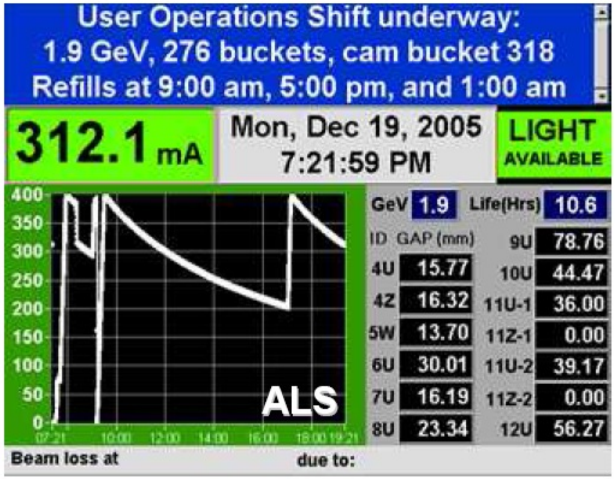
$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3} + ...$$

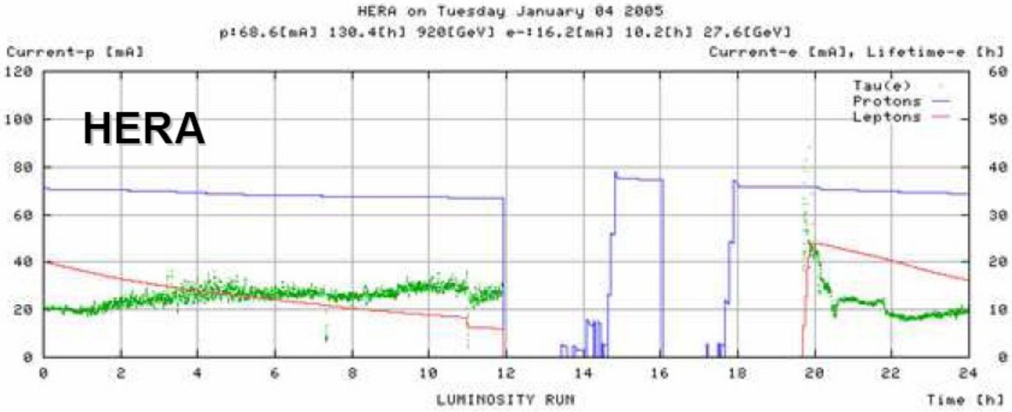
- With this definition, the problem of calculating the lifetime is reduced to the evaluation of the single lifetime components.

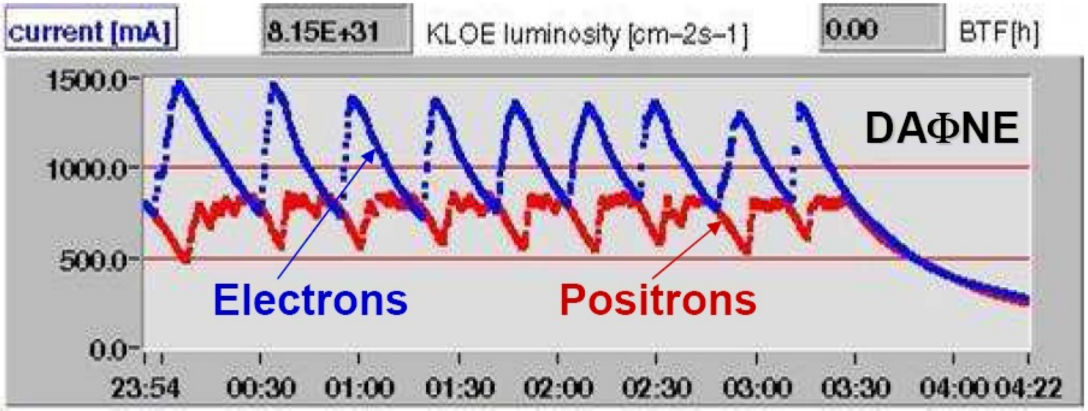
Constant Lifetime?

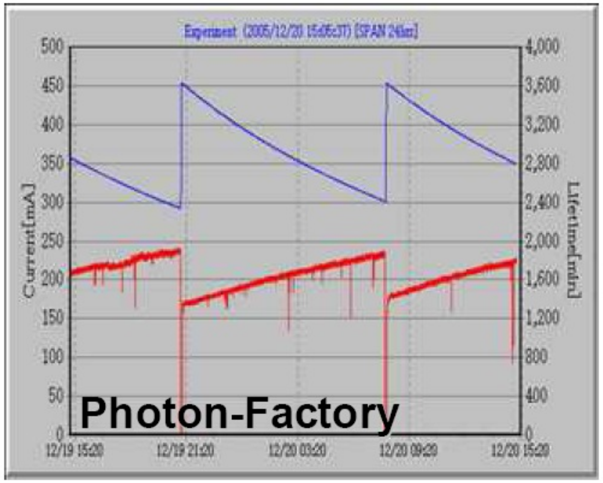
- The previous model assuming constant lifetime is often too simple for describing real accelerators. In most of the electron storage rings the lifetime actually depends significantly on current:
  - The Touschek effect (discussed later), whose contribution dominates the losses in many of the present electron accelerators, depends on current. When the stored current decreases with time, the losses due to Touschek decrease as well and the lifetime increases.
  - Synchrotron radiation intensity and therefore the release of molecules trapped in the vacuum chamber wall depends on current (gas desorption).
    - For higher currents, the pressure in the vacuum chamber increases (dynamic pressure) resulting in more scattering of the beam with the residual gas and a reduced lifetime.
- For reasonably small variations of the current, the constant lifetime assumption is locally valid and it is widely used.

Example Lifetimes in Real Accelerators









Lifetime and beam loss

Why is there a finite lifetime?

- **Electron undergoes a scattering event**
- **Change in angle or energy gives increased amplitude of oscillation**
- **If a boundary is hit (physical or dynamic) then the electron is lost**
- **Gradually all electrons are lost**

Types of scattering

Types of scattering

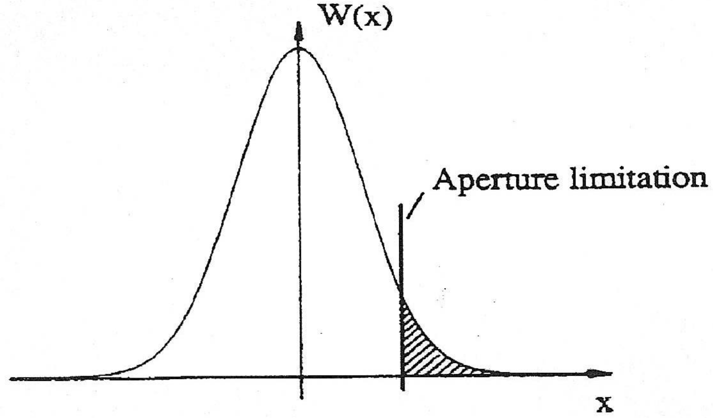
- **Electron-Photon Scattering**
  - **Quantum Lifetime**
- **Electron-Gas Scattering**
  - **Gas Lifetime**
- **Electron-Electron Scattering**
  - **Touschek Lifetime**

Quantum Lifetime

- **Emission of synchrotron radiation is quantized**
- **Distribution of radiation is approximately Gaussian**
- **A Gaussian distribution of particles is produced**
- **Tails of distribution are lost**
- **Redistribution on time scale of damping time**

- **Quantum lifetime is typically more important for colliders than for light sources**

Quantum Lifetime



Beam distribution, W(x)

$$\tau_{Q_T} \cong \tau_{D_T} \frac{\sigma_T^2}{A_T^2} \exp\left(A_T^2/2\sigma_T^2\right) \qquad T = x, y$$

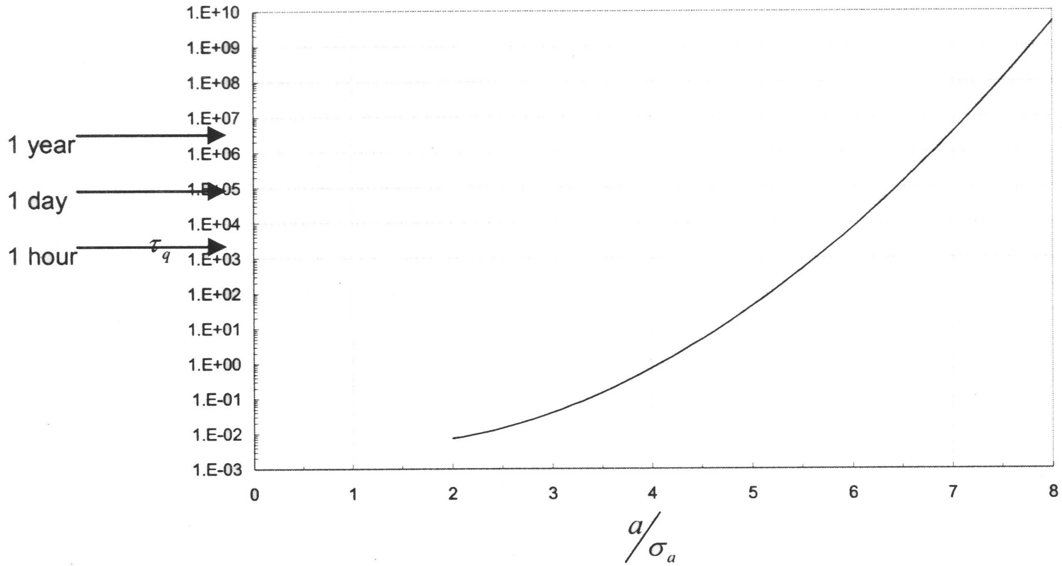
Transverse quantum lifetime

$$\text{where} \quad \sigma_T^2 = \beta_T \varepsilon_T + \left(\eta_T \frac{\sigma_E}{E_0}\right)^2 \qquad T = x, y$$

$$\tau_{Q_L} \cong \tau_{D_L} \exp\left(\Delta E_A^2/2\sigma_E^2\right)$$

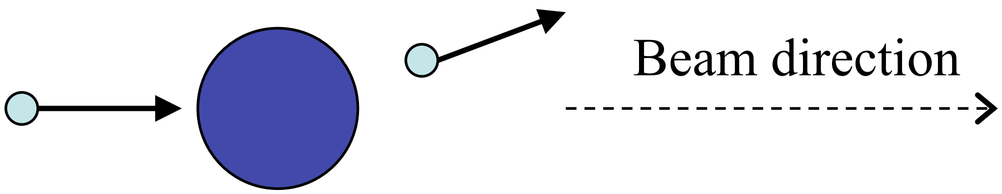
Longitudinal quantum lifetime

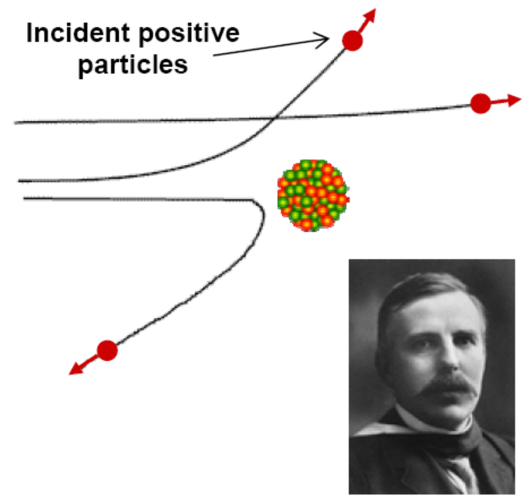
Quantum Lifetime versus aperture-to-beamsize ratio



Gas-scattering lifetime

Particles scatter elastically or inelastic with residual gas atoms. This introduces betatron or synchrotron oscillations.





The scattering process can be described by the classical Rutherford scattering with differential cross section per atom in cgs units

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{2\beta cp}\right)\frac{1}{\sin^4\frac{\theta}{2}}$$

Gas-scattering lifetime

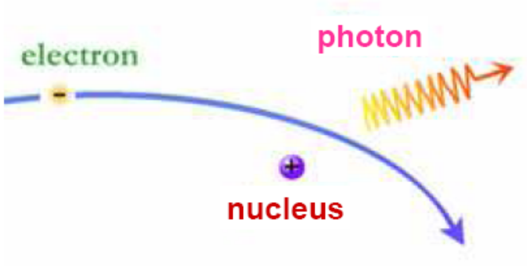
If the new amplitudes are outside the aperture the particles are lost.

•The elastic scattering lifetime is proportional to the square of the transverse aperture *A*:

$$\frac{1}{\tau_{el}} \propto \frac{1}{E^2} \times \left(\frac{\beta_x}{A_x^2} \langle P\beta_x \rangle + \frac{\beta_y}{A_y^2} \langle P\beta_y \rangle \right)$$

• The inelastic scattering lifetime is proportional to the logarithm of the energy/momentum aperture ε:

$$\frac{1}{\tau_{inel}} \propto \langle P \rangle \times \ln(\varepsilon)$$



Gas Lifetime – Vacuum Requirements

- For electrons one can simplify the formulas for gas Bremsstrahlung lifetime (in the approximation of <Z^2> ~ 50):

$$\tau_{Brem[hours]} \cong -\frac{153.14}{\ln\left(\Delta E_A / E_0\right)} \frac{1}{P_{[nTorr]}}$$

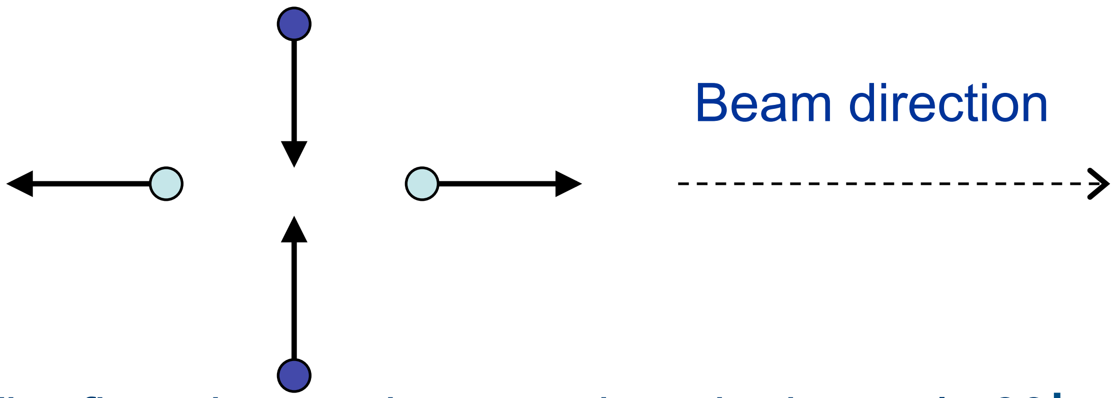
- In the same approximation, the elastic gas scattering lifetime becomes:

$$\tau_{Gas[hours]} \cong 10.25 \frac{E^2_{0[GeV]}}{P_{[nTorr]}} \frac{\varepsilon_{A[\mu m]}}{\langle \beta_T \rangle_{[m]}}$$

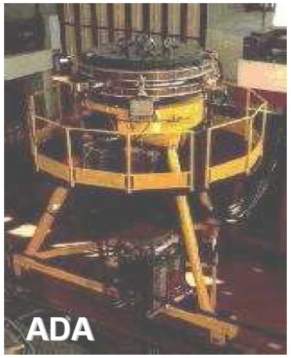
- For typical electron ring parameters, one finds that the requirement on vacuum is for dynamic pressures of the order of a few nTorr.

Touschek Lifetime

Particles inside a bunch perform transverse betatron oscillations around the closed orbit. If two particles scatter they can transform their transverse momenta into longitudinal momenta.



- The first observation was done in the early 60's in Frascati at ADA, the electron-positron accelerator conceived by the Austrian scientist Touschek.
- The Touschek effect is the dominant lifetime contribution in many modern electrons storage rings.



Touschek Scattering

- **Large angle electron-electron scattering**
  → single scattering leads to loss
- **Calculate scattering cross-section: Møller cross section, which reduces to** $d\sigma \; \mathrm{p} \; \frac{1}{\beta^2}\left(\frac{1}{\sin^4\theta} - \frac{1}{\sin^2\theta}\right) d\Omega$
- **Above formula is correct for non relativistic velocities (in restframe of particle bunch) and if there is no average polarization**
  - **In reality effect of polarization not negligible (see my talk on energy calibration)**
- **If the new momenta of the two particles are outside the momentum aperture,** ε **, the particles are lost. The lifetime is proportional to the square of** ε

$$\frac{1}{\tau_{tou}} \propto \frac{1}{E^3} \frac{I_{bunch}}{V_{bunch}\sigma_x'} \frac{1}{\varepsilon^2} f\left(\varepsilon, \sigma_x', E\right)$$

Lifetime Limiting Processes

- Elastic Scattering $$\frac{1}{\tau_{el}} \propto \frac{1}{E^2} \times \left(\frac{\beta_x}{\Delta_x^2} \langle P\beta_x \rangle + \frac{\beta_y}{\Delta_y^2} \langle P\beta_y \rangle \right) \qquad (1)$$
- Touschek Effect $$\frac{1}{\tau_{tou}} \propto \frac{1}{E^3} \frac{I_{bunch}}{V_{bunch} \sigma'_x} \frac{1}{\varepsilon} f\left(\varepsilon, \sigma'_x, E\right) \qquad (2)$$
- Quantum Lifetime $$\frac{1}{\tau_q} \propto \frac{\Delta^2}{\sigma^2} \times \exp\left(-\frac{\Delta^2}{2\sigma^2} \right) \qquad (3)$$
- Inelastic Scattering $$\frac{1}{\tau_{inel}} \propto \langle P \rangle \times \ln(\varepsilon) \qquad (4)$$

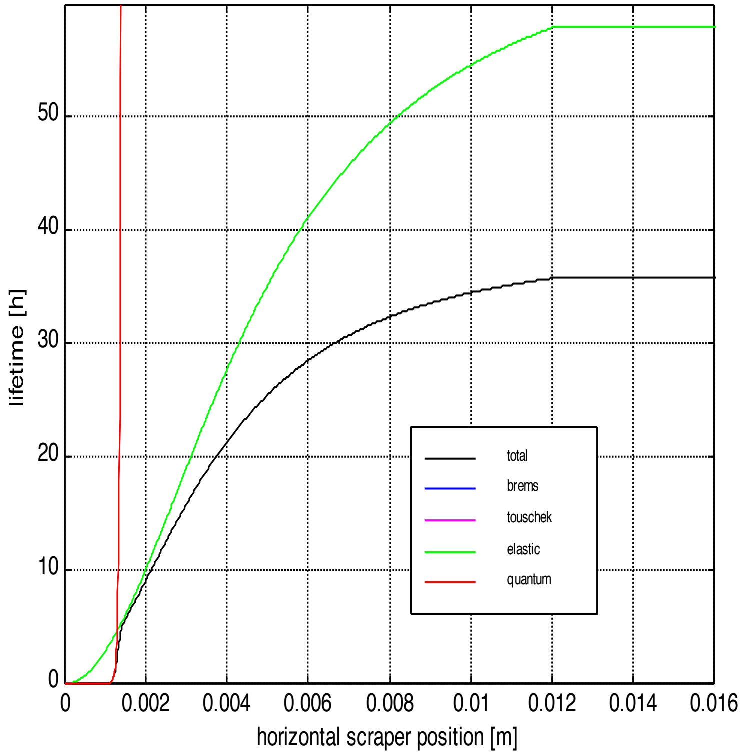
$$\frac{1}{\tau} = \frac{1}{\tau_{el}} + \frac{1}{\tau_{tou}} + \frac{1}{\tau_{ql}} + \frac{1}{\tau_{inell}}$$

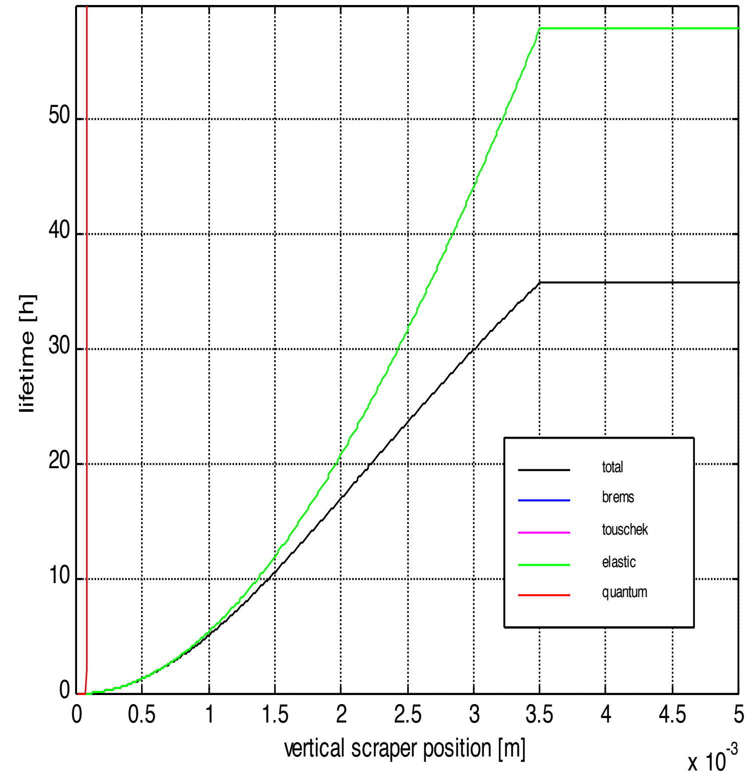
Lifetime in different accelerator types

- In proton and heavy ions storage rings no damping is present: Any perturbation can build up and can eventually lead to particle loss.
- However, important electron loss mechanisms are negligible for protons. These include for example, Touschek and gas bremsstrahlung scattering as well as the quantum lifetime.
- Other effects such as *elastic gas scattering*, *molecule excitation*, *fluctuations in the magnetic and RF fields*, *Coulomb scattering (intra-beam scattering)*, …, add up to generate a lifetime of the order of typically hundreds of hours compared to lifetimes of maybe a few hours in synchrotron light sources.
- Quite often in colliders, the interaction between the colliding beams, the so-called *beam-beam effect*, becomes the main mechanism of losses.
- In the following slides I will study the case of light sources more closely.

Dependency of Lifetime on Transverse Aperture

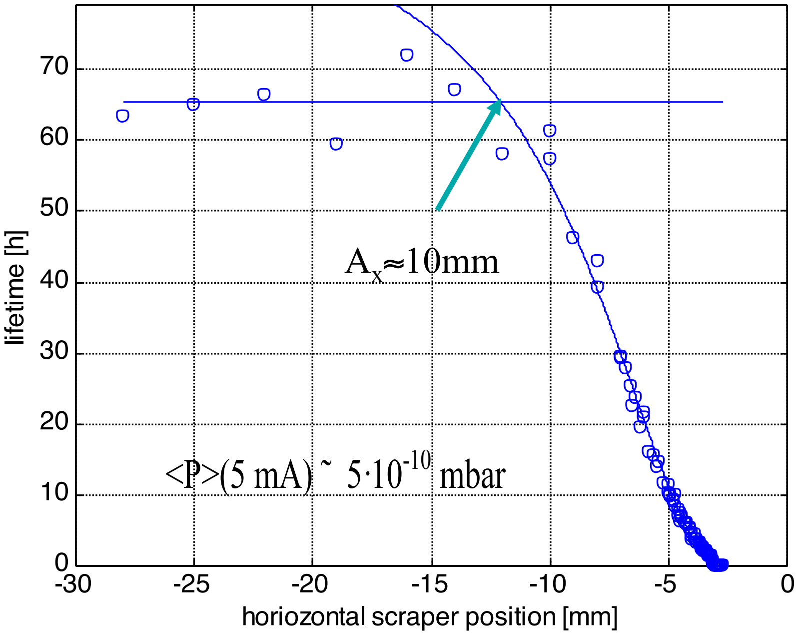
Theoretical Results

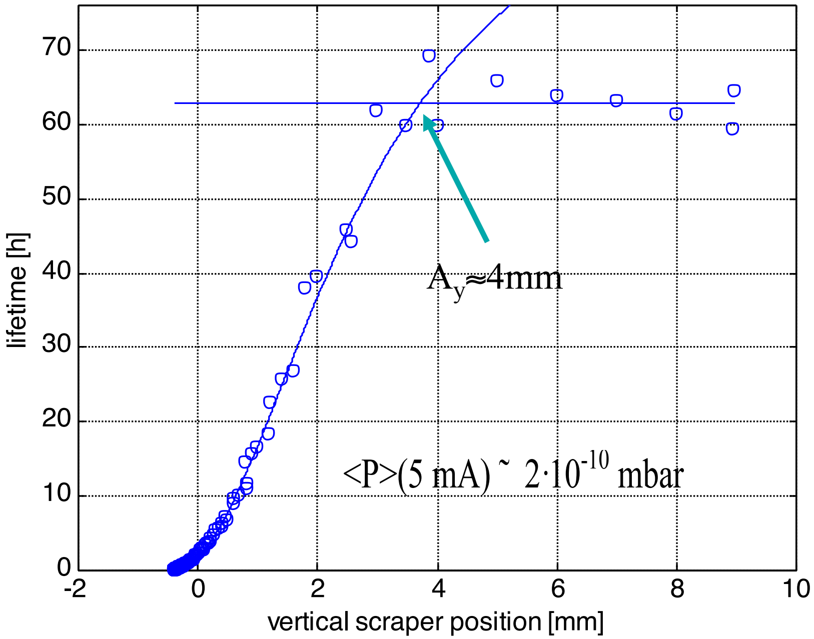




ALS Transverse Acceptance and Gas Lifetime

- move scraper into beam and record lifetime: acceptance, gas pressure





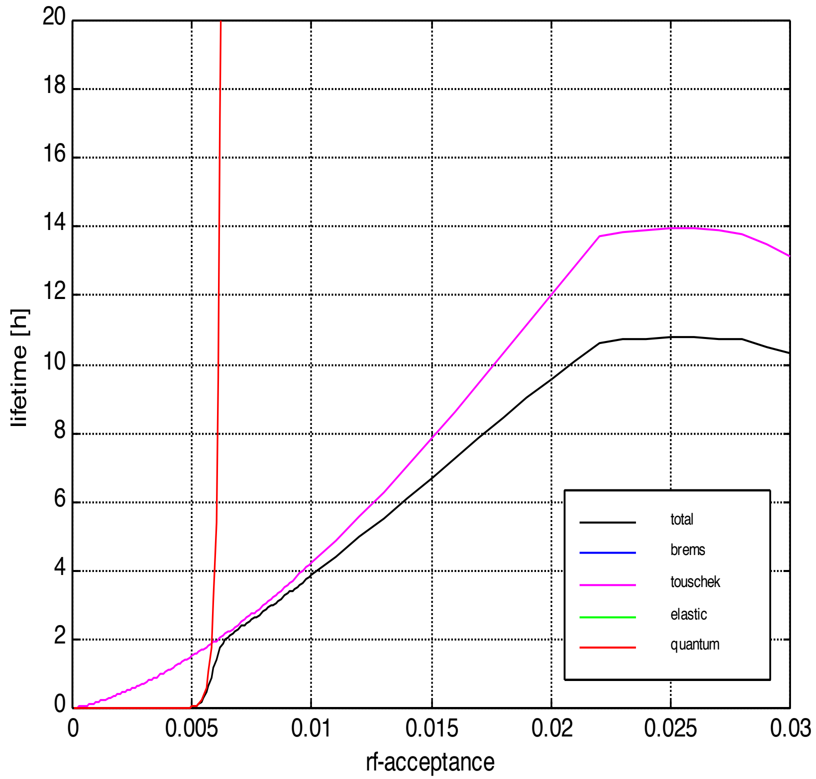
$$\frac{1}{\tau}(\Delta_x) = \begin{cases} const. & \text{if } \Delta_x > A_x \\ \frac{1}{\tau_{tou+inel}} + C_{el}\frac{1}{E^2}\langle P\rangle\left(\langle\beta_x\rangle\frac{\beta_x}{\Delta_x^2} + \langle\beta_y\rangle\frac{\beta_y}{A_y^2}\right) & \text{if } \Delta_x < A_x \end{cases}$$

Assuming different distribution of the gas, i.e. higher pressure in the straight sections: **$3*10^{-10}$ mbar**

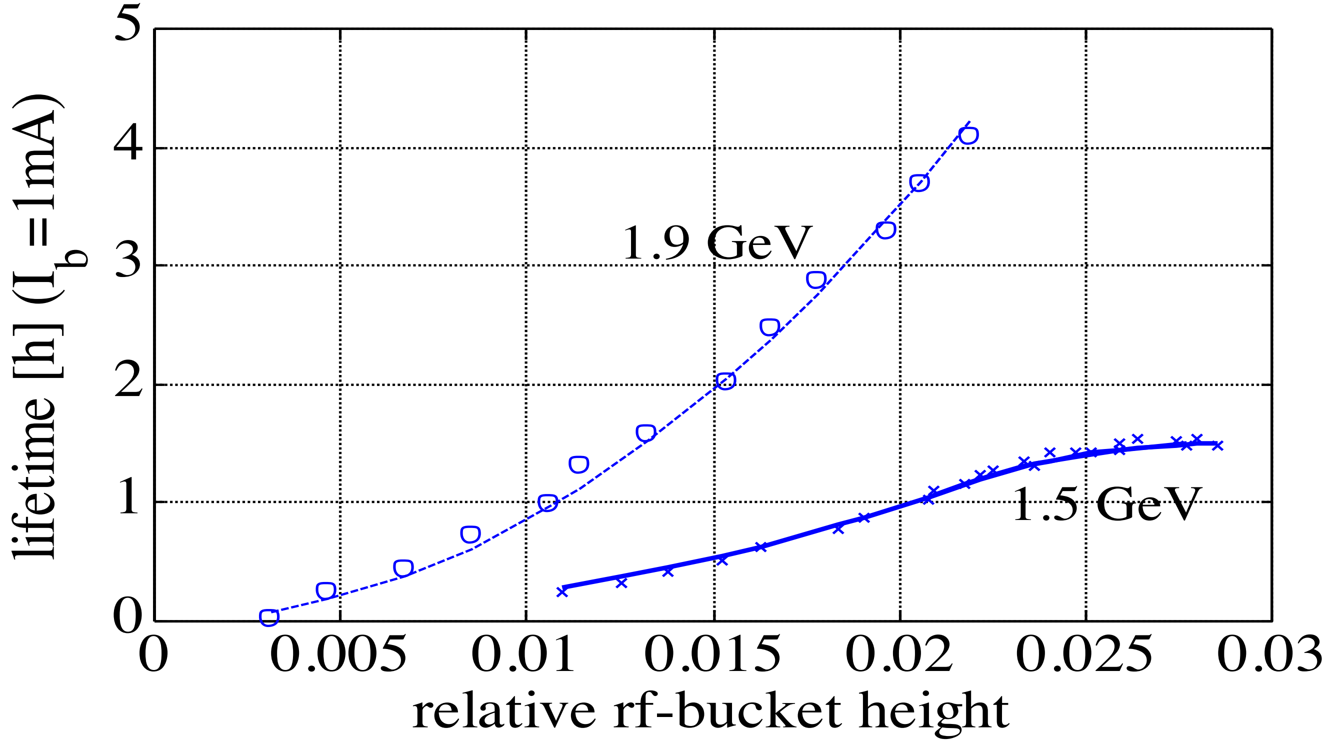
Desorption coefficient: $1.75*10^{-12}$ mbar/mA

Dependency of Lifetime on Longitudinal Aperture

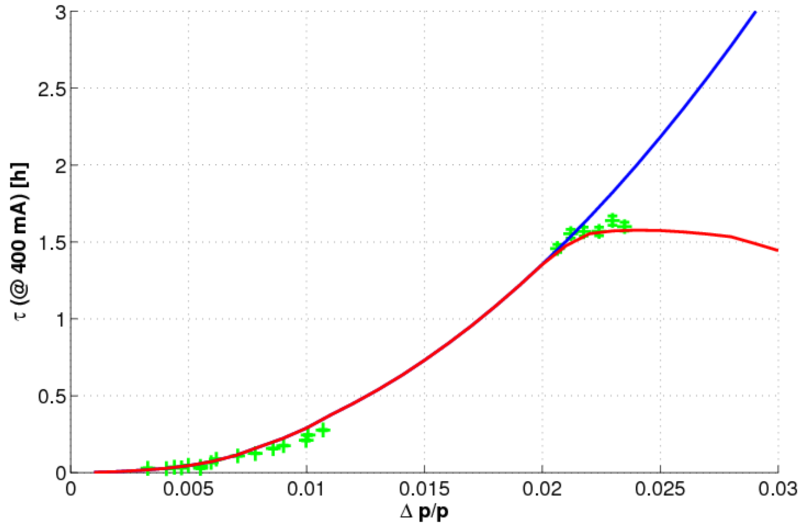
Theoretical results including bunch length change



Lifetime versus RF-Bucket Height



Touschek lifetime scans



- Calculate RF voltage dependent Touschek lifetime based on calibrated machine model (emittance, beamsized, lattice function, s-position dependent dynamic momentum aperture all calculated from calibrated model)

- Compare measurements (green errorbars) with those calibrated calculation
  - Excellent Agreement

Off-energy dynamics: Touschek Lifetime

- Lifetime is crucial performance parameter for light sources ⇒ for 3rd generation light sources limit is Touschek lifetime ⇒ strong function of momentum aperture ε

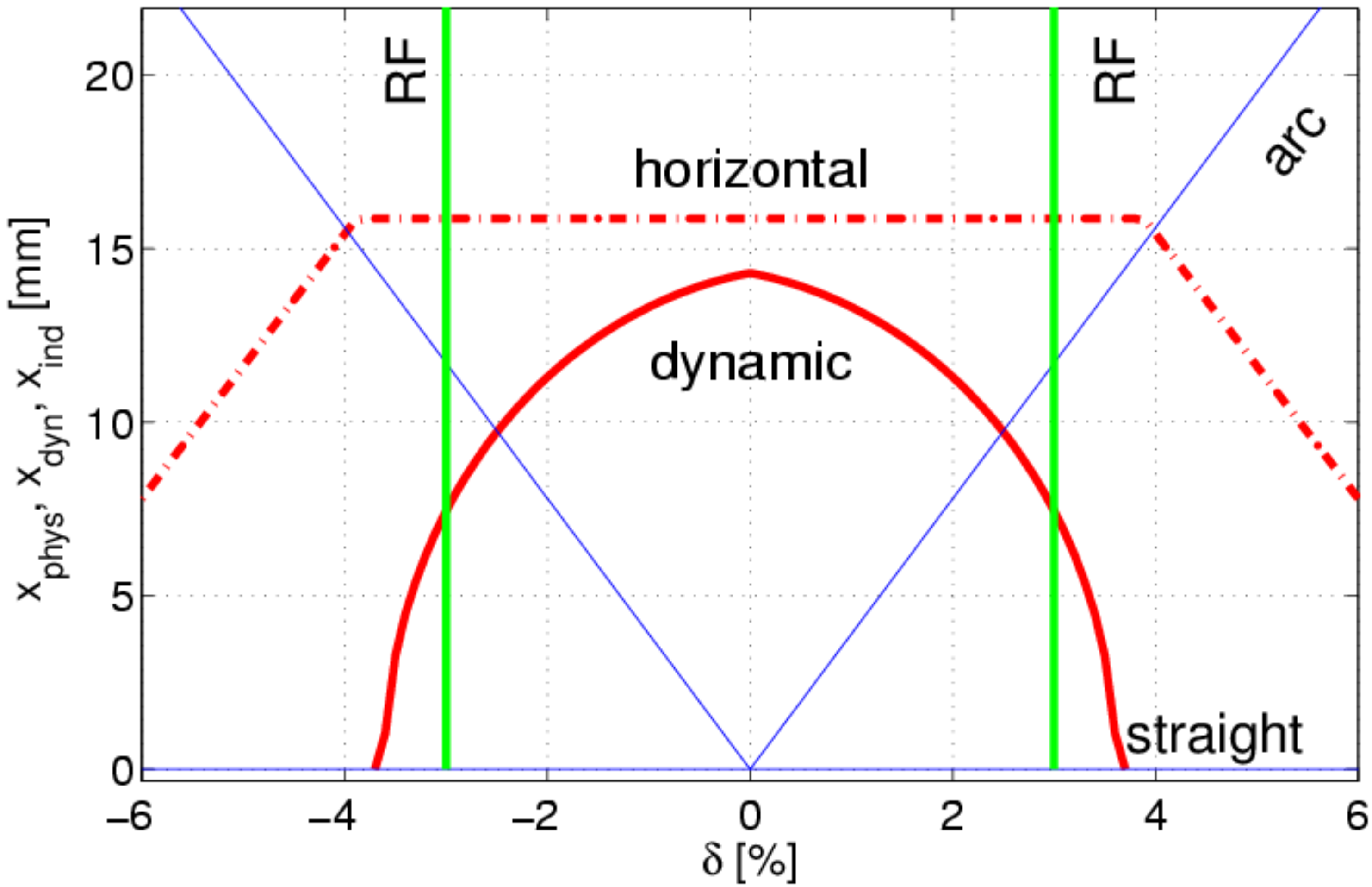
$$\frac{1}{\tau_{tou}} \propto \frac{1}{E^3} \frac{I_{bunch}}{V_{bunch}\sigma_x'} \frac{1}{\varepsilon^2} f\left(\varepsilon, \sigma_x', E\right)$$

- Momentum aperture ε is often limited by single particle dynamics
- 3rd generation light sources with their strong focusing to achieve small equilibrium emittances (small dispersion) and very strong sextupoles did originally not achieve their design momentum apertures of about 3%.

Motivation for off-energy dynamics studies

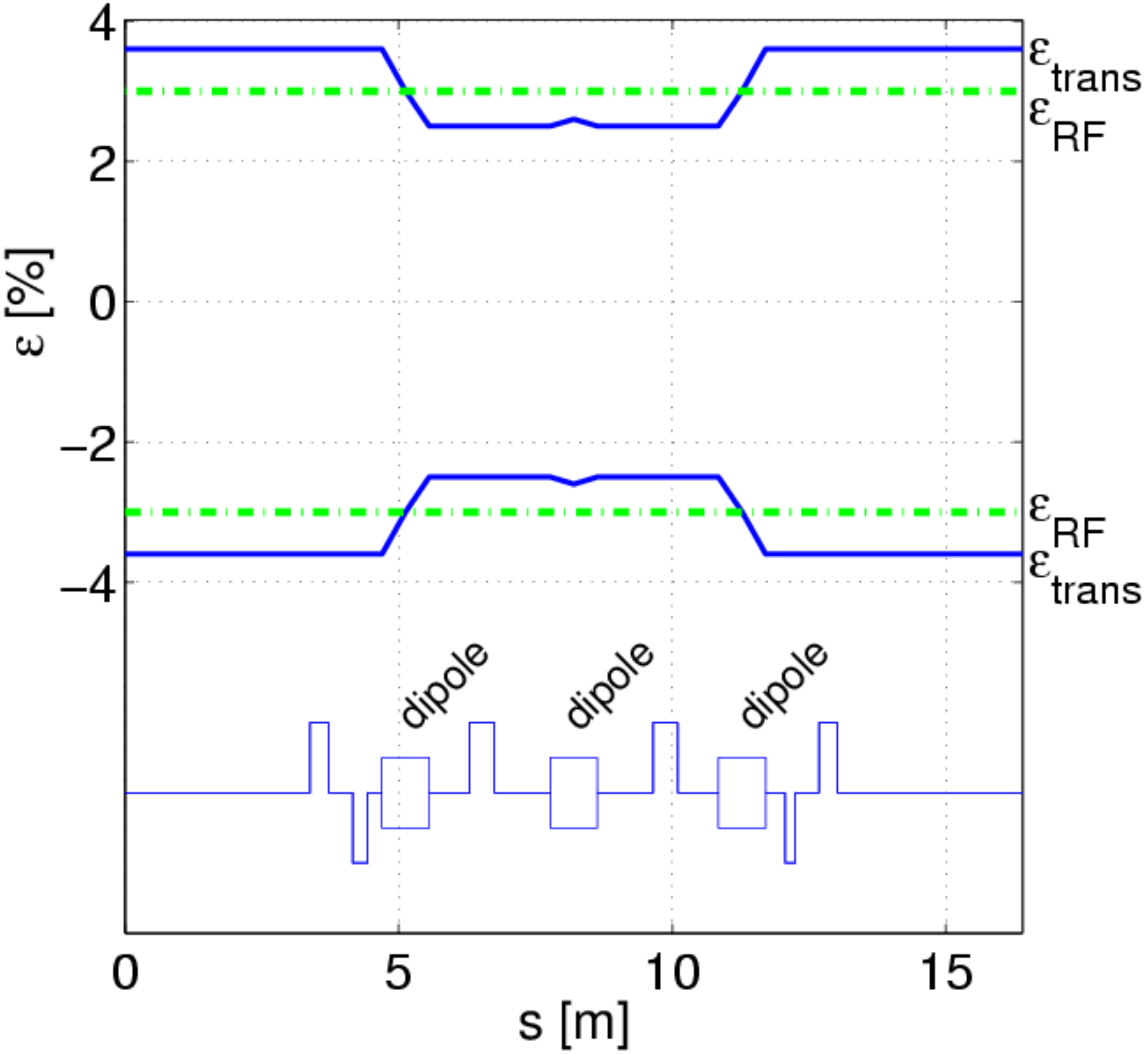
- Design momentum aperture for newer light sources (like Soleil) 5-6% to achieve reasonable lifetimes
- Even using top-up (quasi continous) injection, lifetime is still an issue:
  - Radiation damage/safety
  - Injection transients are not fully transparent

What determines the momentum aperture



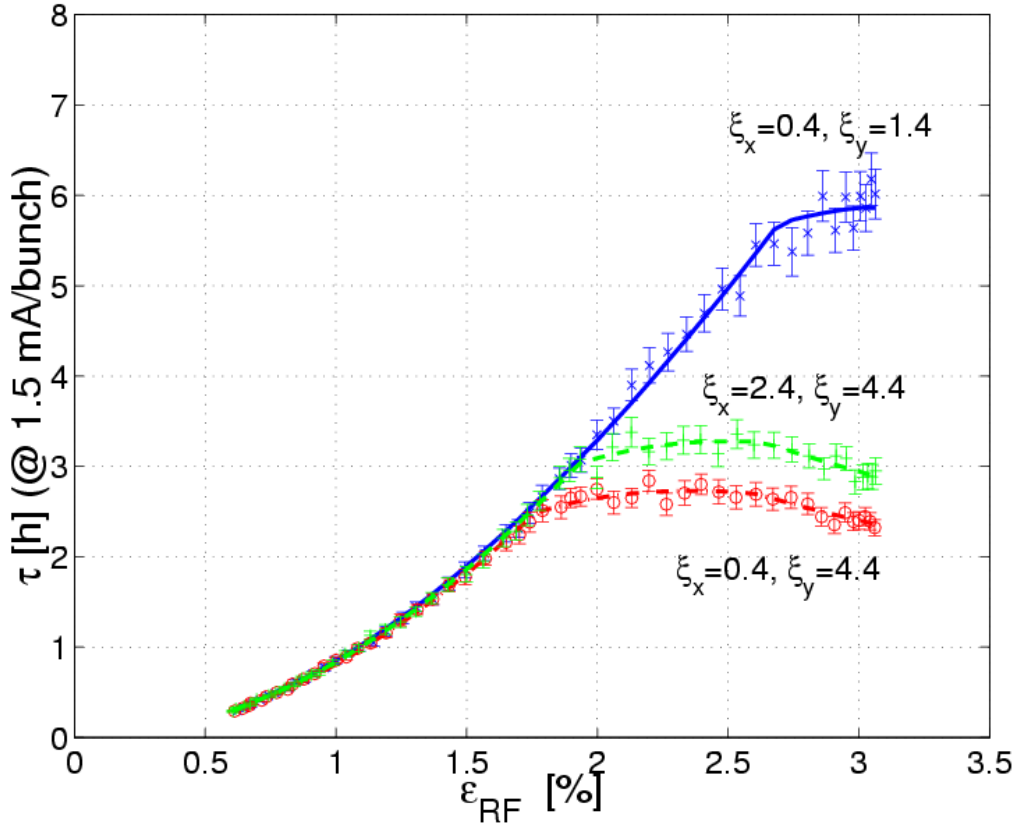
- Possible to quantitatively measure momentum aperture by scanning RF voltage

Longitudinal variation of momentum aperture



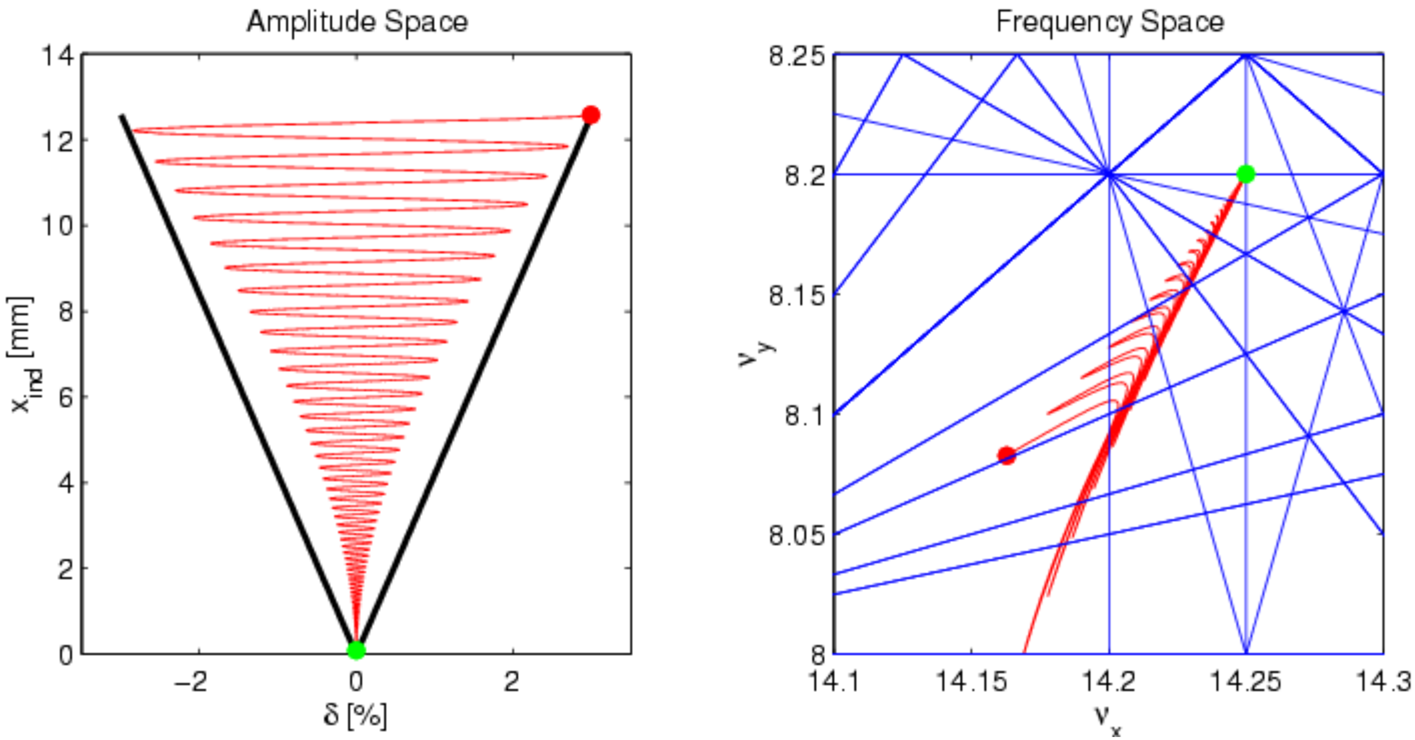
- Because of variation in H-function, momentum aperture will vary around the ring (depending on scattering location)
- Not necessarily symmetric for positive and negative momentum deviation (asymmetric bucket)

ALS example: RF amplitude



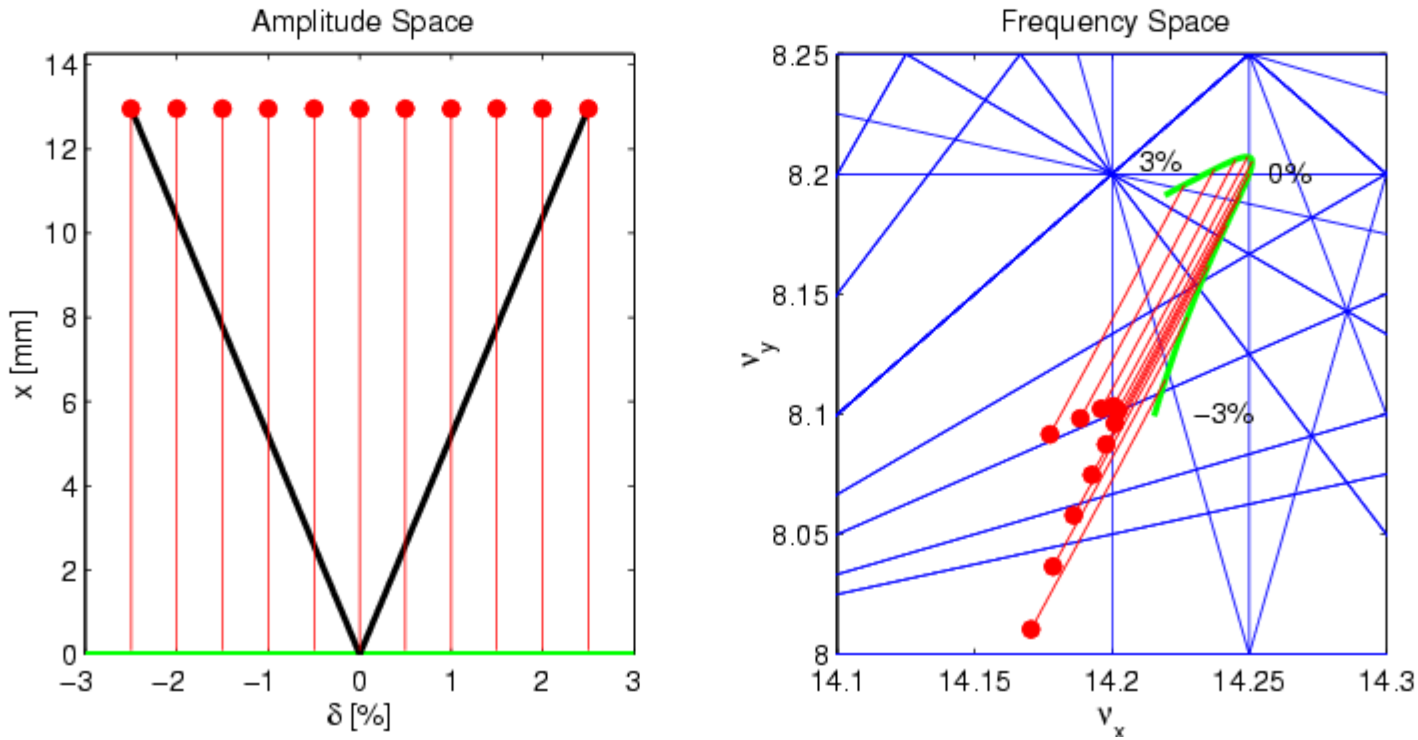
- Momentum aperture in ALS is clearly impacted by dynamics
- Sensitivity to chromaticity is at first surprisingly large (sextupole strength only different by a few percent).

Touschek Scattering – Tune Shift – Particle Loss



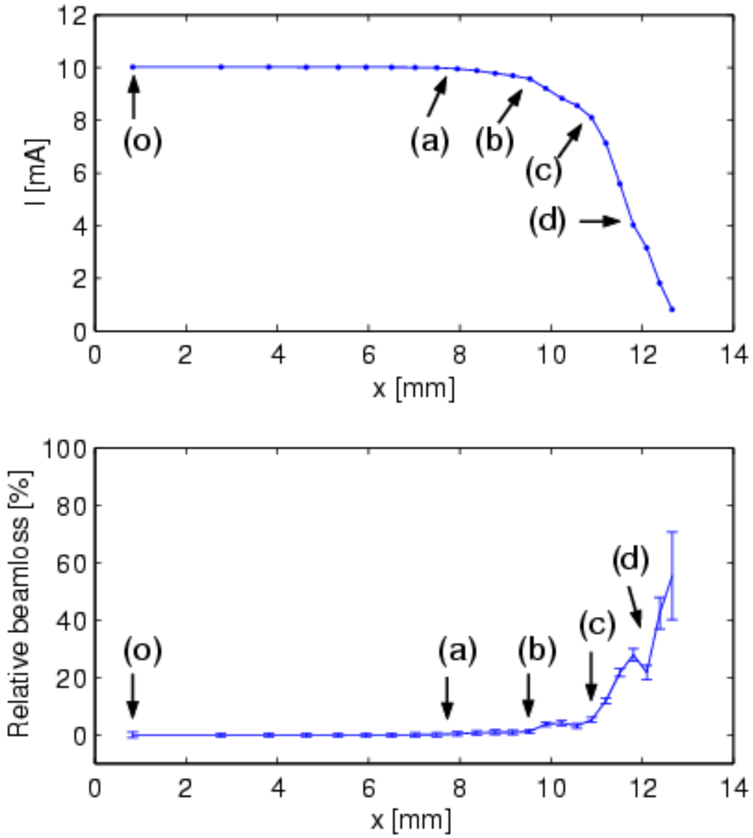
- Particle losing/gaining energy – horiz. oscillation (dispersion/H-function) + long. Oscillation
- Particle changes tune
  - Synchrotron oscillations (chromaticity)
  - Radiation damping (detuning with amplitude and chromaticity)
- During damping process particle can encounter region in tune space where motion gets resonantly excited.

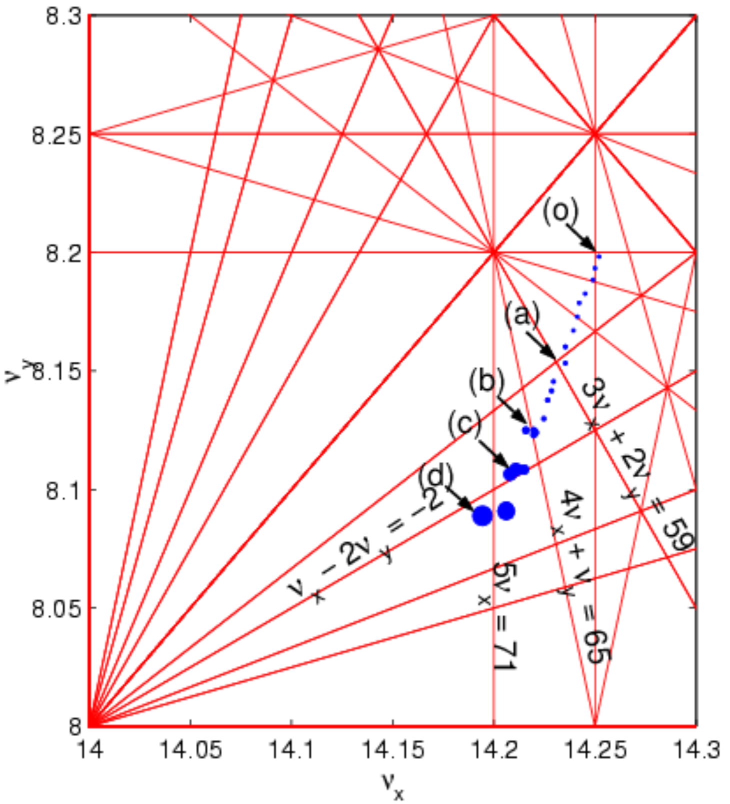
Measurement principle



- **Experimentally very difficult to exactly simulate Touschek scattering (simultanous kicks) – also difficult to measure tunes during synchrotron oscillations**
  - **Some positive results (Y. Papaphilippou et al.)**
- **Still possible to locate loss regions when scanning only transverse amplitude while keeping energy offset fixed**

Measurement Detail

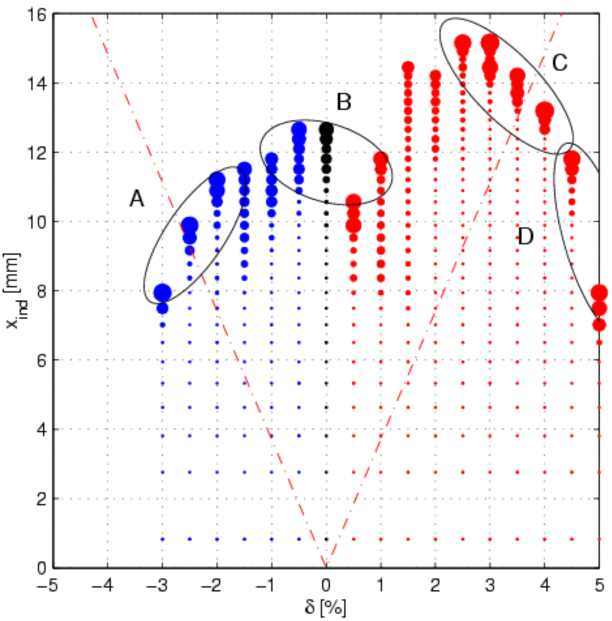




- Use single turn kicker to excite beam with increasing amplitude
- Use current monitor to record relative beam loss after kick
- Use turn-by-turn BPMs to record oscillation ferquencies

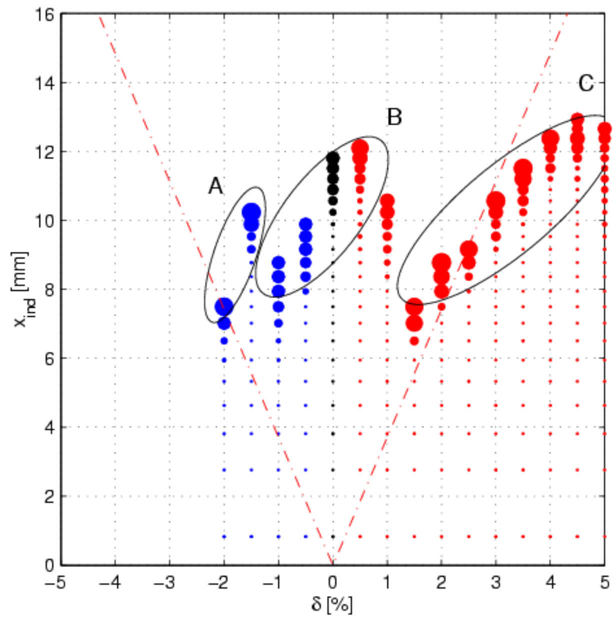
Aperture Scan for 3 Different Chromaticities

Small horiz. Chromaticity
Small vert.



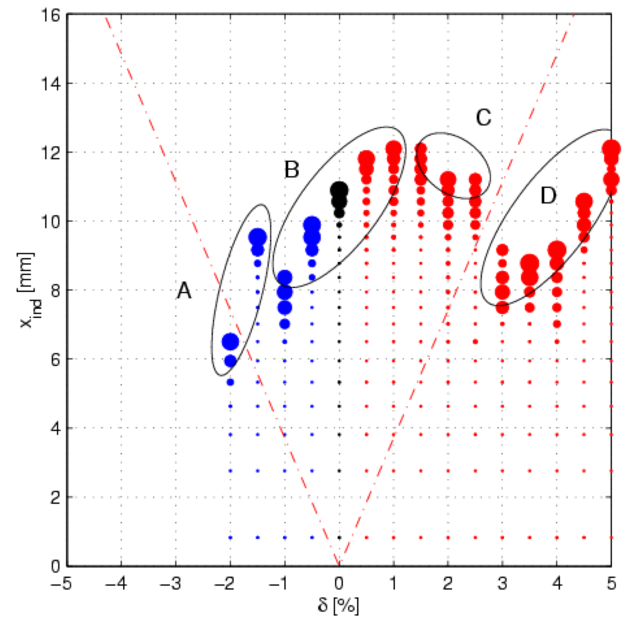
ε > 3 % straight
2.65 % arcs

Small horiz.
Large vert.



ε = 2.6 % straight
1.75 % arcs

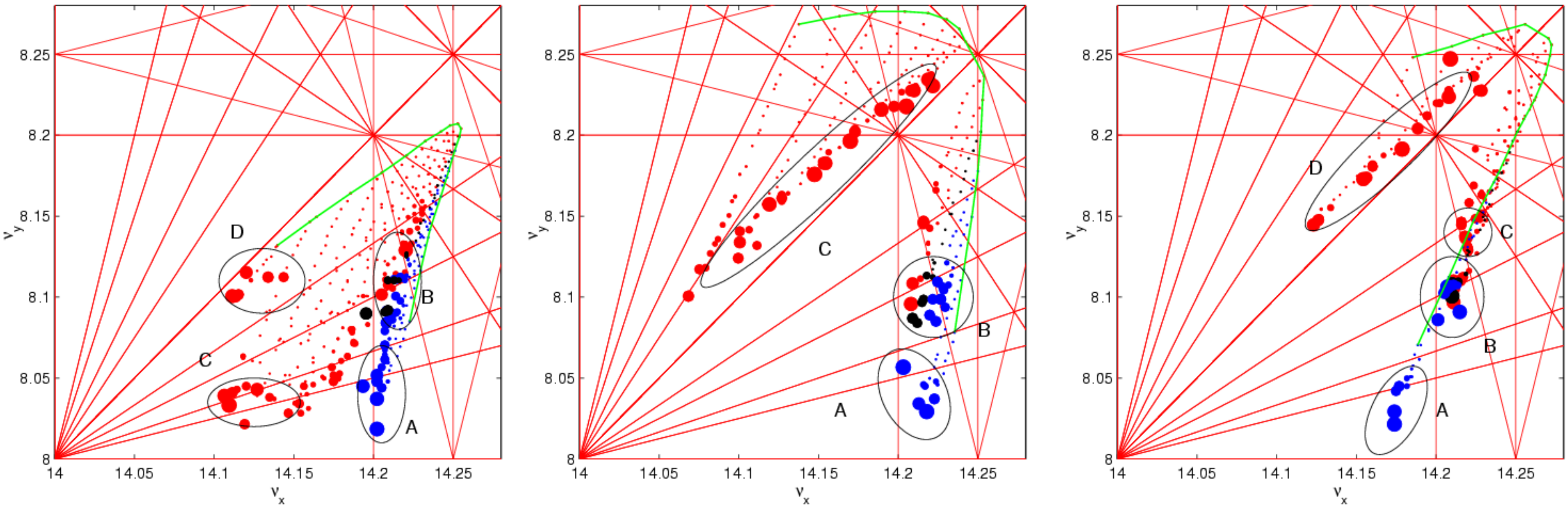
Large horiz.
Large vert.



ε = 2.6 % straight
1.9 % arcs

Aperture Scan for 3 Different Chromaticities

Small horiz. Chromaticity Small vert. | **Small horiz. Large vert.** | **Large horiz. Large vert.**

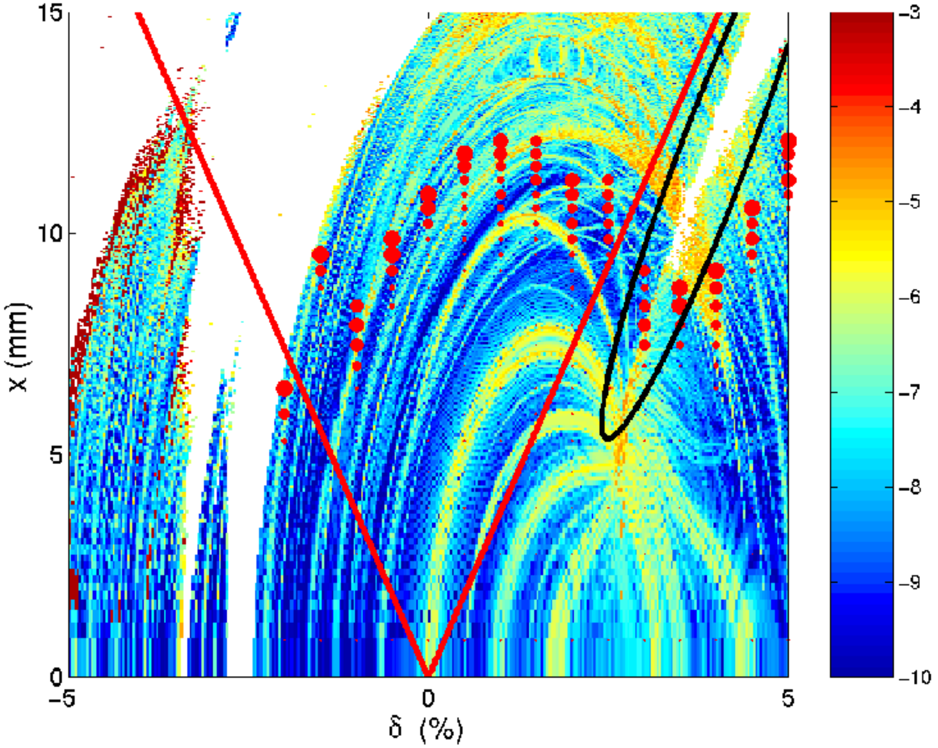


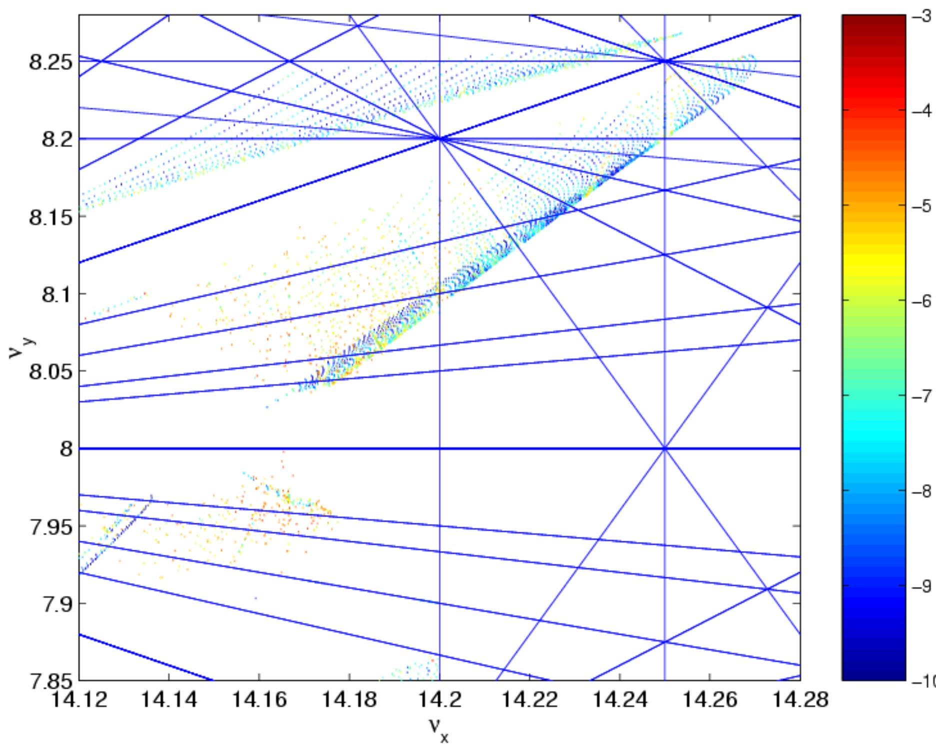
ε > 3 % straight
2.65 % arcs

ε = 2.6 % straight
1.75 % arcs

ε = 2.6 % straight
1.9 % arcs

Results agree well with Simulations



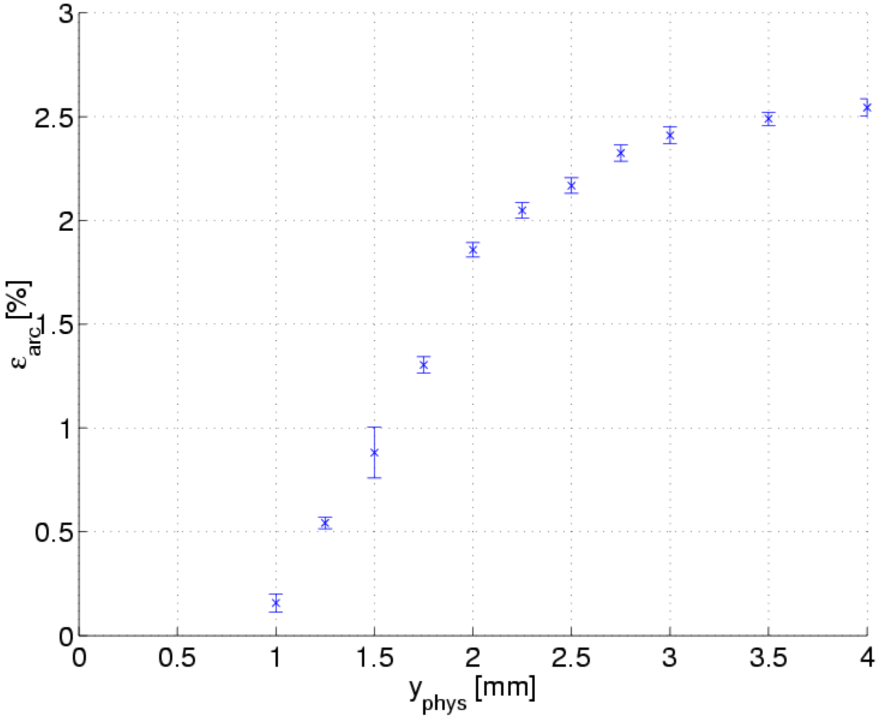


- Simulations reproduce shift of beam loss area caused by the coupling resonance to higher momentum deviations

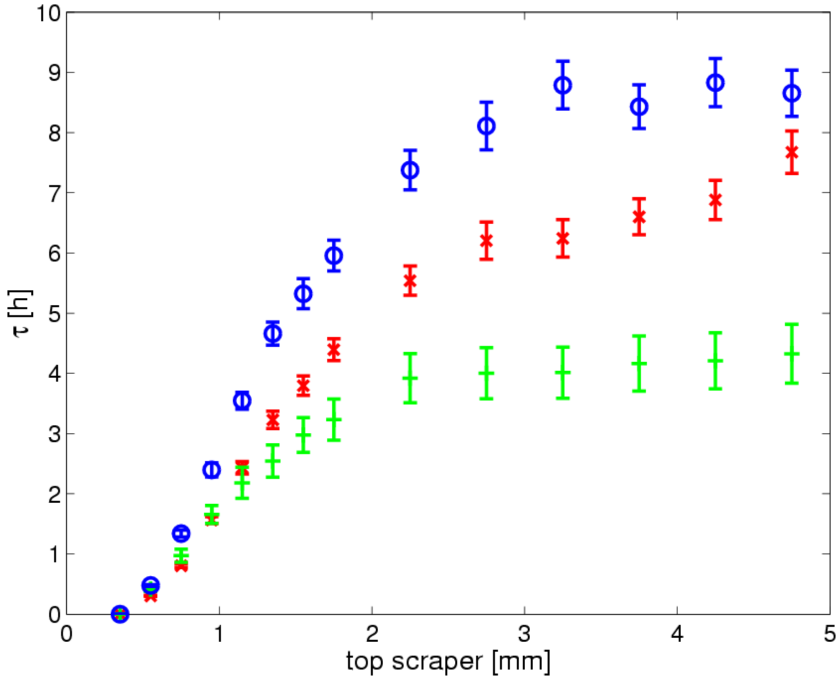
Lifetime vs. Vertical Physical Aperture

- Found dynamic+momentum 'aperture' and (small) vertical physical aperture very closely linked – for ALS momentum aperture collapses around 40-50 σ_y
- Since both are very important performance parameters studied link further:

- Performance (Brightness) of undulators/wigglers (both permanent magnet and SC) depends on magnetic gap
- Strong incentive to push physical aperture as low as possible
- Evolution at the ALS from 15 mm via 9 and 8 mm to now 5.5 mm – enabled by better understanding and optimization



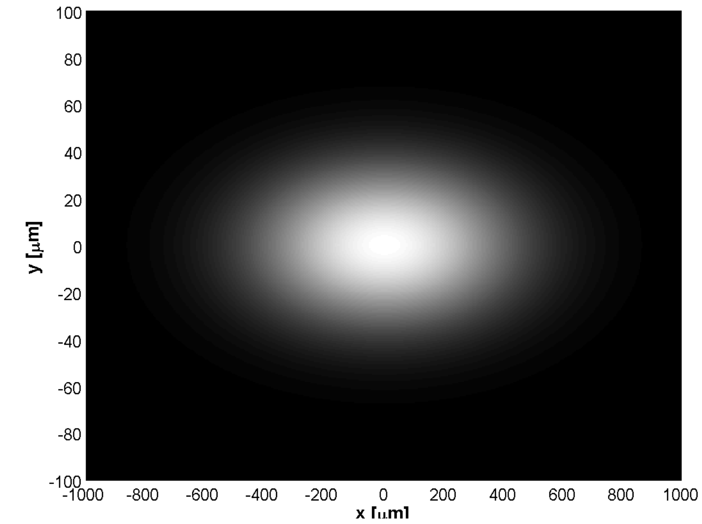
Coupling – Sensitivity to Physical Aperture

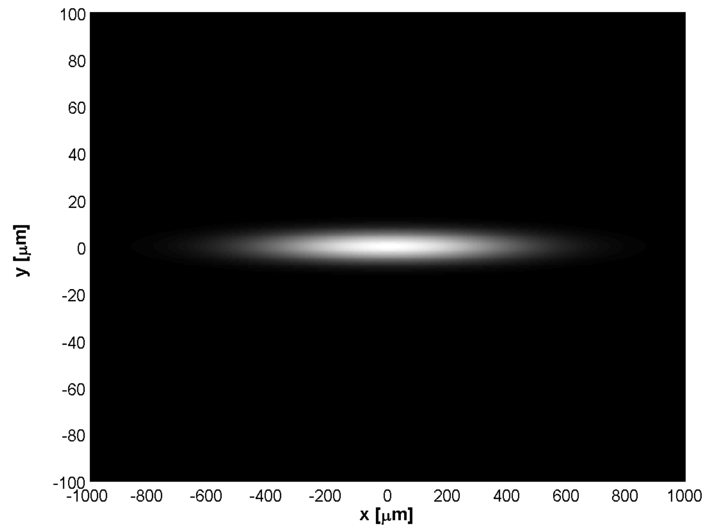


- Sensitivity of Touschek lifetime on vertical aperture depends on coupling
- High order coupling resonances scale similar to global/local coupling
- For given emittance ratio one can optimize coupling vs. vertical dispersion

Ultrasmall coupling allows small gaps

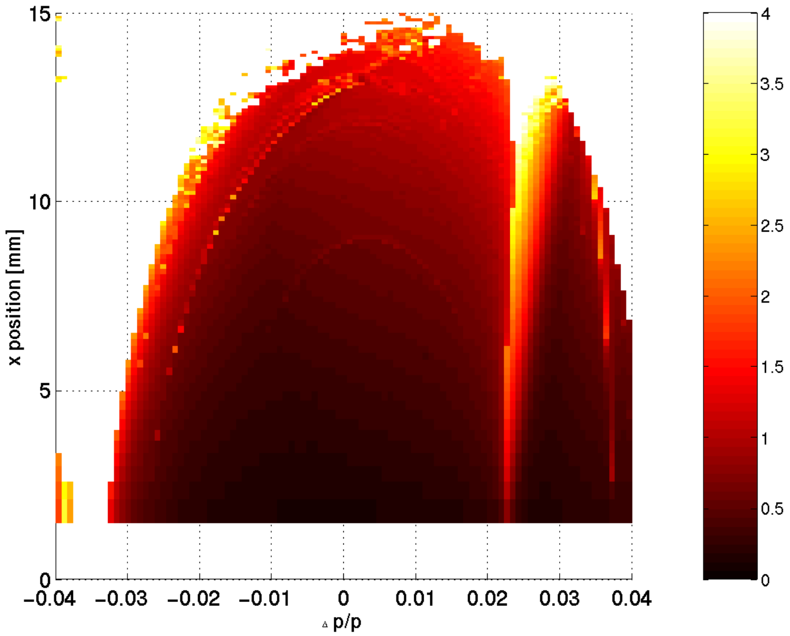
- Using LOCO and an optimized (based on simulations) skew quadrupole distribution with 18 skew quadrupole we achieved an emittance reduction from 150 pm (routine ALS operation) to about 5 pm (pictures on the right illustrate size reduction for insertion device straights)
- This was a world record at the time (now ATF about 4 pm) and about the NLC damping ring design value
- Correcting to those small coupling values and then using coupling free dispersion wave allows smaller vertical physical apertures.

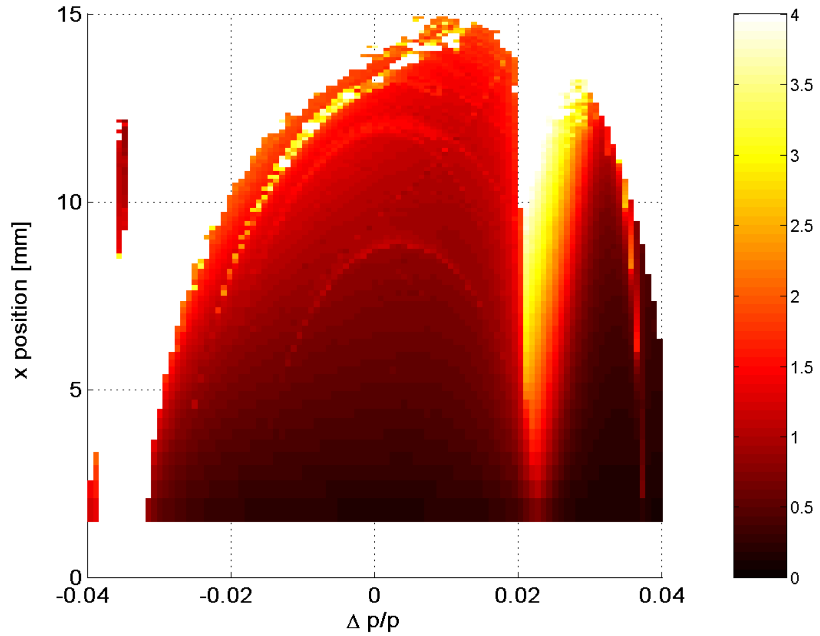




Simulation Results (Momentum Aperture – Gap)

Emittance increased using vertical dispersion wave … using excitation of coupling resonance





- Tracking results are in good agreement with measured effects, i.e. case with dispersion wave has less yellow and orange areas than the one with excited coupling resonance, indicating less sensitivity to reduced vertical aperture

Summary

- Beam lifetime is determined by various scattering processes
- Can use scrapers, loss monitors, turn-by-turn BPMs, … to study the details of lifetime limiting processes
- For many machines, the transverse single particle dynamics plays an important role for the injection efficiency (dynamic aperture) and (Touschek) lifetime (momentum aperture)
  – Measurement method using frequency analysis provides a very powerful model independent tool
  – Method has large potential in new light sources with many sextupole families
- Agreement between measurements and simulations are very good

Further Reading

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