

USPAS 2015: East Brunswick, Rutgers

**Transverse and Longitudinal Beam
Dynamics Fundamentals**

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Outline

- Motivation
- Transverse Beam Dynamics
 - Hill's Equation
 - Twiss Functions (Beta-Function, ...)
 - Tune, Resonances
 - Emittance
 - Matrix Formalism
 - Basis for simulation codes
- Longitudinal Dynamics
 - Time of Flight, Synchrotron Oscillations
- Radiation
 - Damping/Excitation, Equilibrium Emittances

http://als.lbl.gov/als_physics/csteier/uspas15/

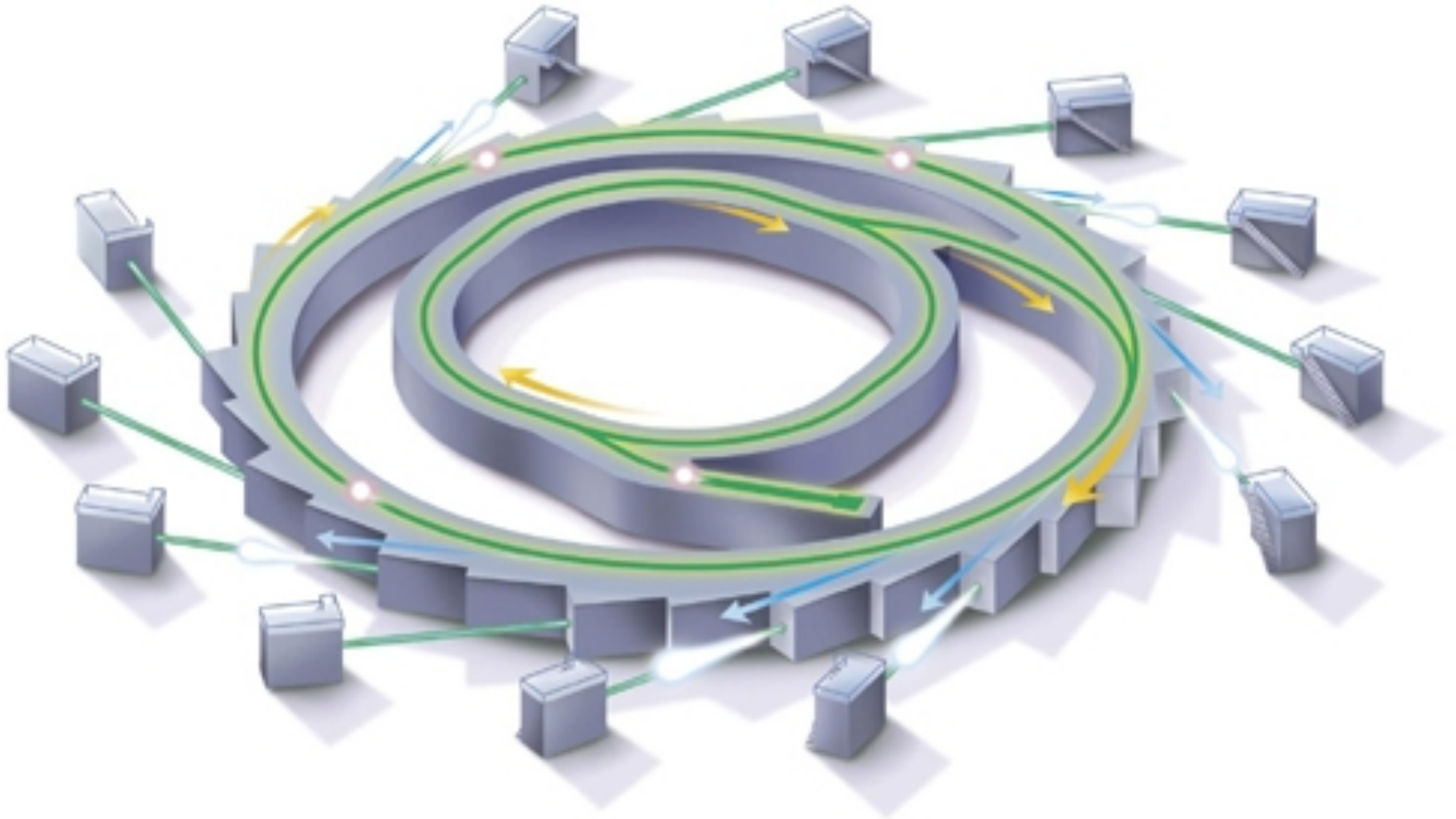
Motivation

- This course will deal in detail with measurements involving many areas of transverse (and longitudinal) single (and multiple) particle dynamics
- Most (but not all) of you already have learned all fundamentals
- Still would like to remind you of all concepts, to get all to somewhat consistent starting point for class
 - For transverse dynamics will introduce lattice functions in two different ways (including the one usually used in lattice codes, which you might not have learned, yet).
- *No need to get scared by this pretty dense lecture. The remainder of the week will be much more practical – and does not require that you completely understand everything in this recap*
- *Disclaimer: Our class is storage ring biased. Basic concepts and measurements are applicable to transfer lines and linacs, but details are different. If you have questions regarding lines, linacs, protons: You are welcome to ask at any time.*

Transverse Beam-dynamics: Terminology

- Linear beam dynamics (Today) determined by:
 - Dipoles
 - Quadrupoles (lenses)
 - Solenoids
 - RF-resonators
 - (synchrotron radiation)
- Nonlinear (Wednesday/Thursday):
 - Sextupoles, higher multipoles, errors, insertion devices (undulators/wigglers), stochastic nature of SR, ...
- Trajectory/Orbit – (more on closed orbit Tomorrow)
 - Closed orbit: closed, periodic trajectory around a ring (closes after one turn in position and angle).
 - Particles that deviate from the closed orbit will oscillate about it (transverse: Betatron oscillations, longitudinal: Synchrotron Oscillations)

ALS Main 'ingredients' of an Accelerator



Electromagnetic Fields

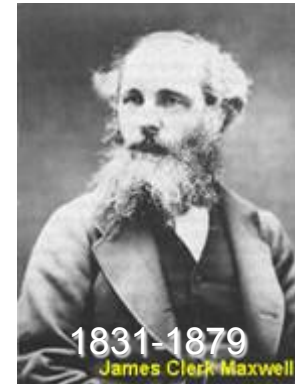
Maxwell Equations in vacuum (SI Units – differential form):

$$\nabla \cdot \bar{E} = \frac{\rho}{\epsilon_0} \quad \text{Coulomb's or Gauss' law for electricity}$$

$$\nabla \times \bar{E} = - \frac{d\bar{B}}{dt} \quad \text{Faraday's law}$$

$$\nabla \cdot \bar{B} = 0 \quad \text{Gauss' law for magnetism}$$

$$\nabla \times \bar{B} = \mu_0 \bar{J} + \mu_0 \epsilon_0 \frac{d\bar{E}}{dt} \quad \text{Ampere's law}$$



Time variable magnetic fields are always associated with electric fields (and vice versa)

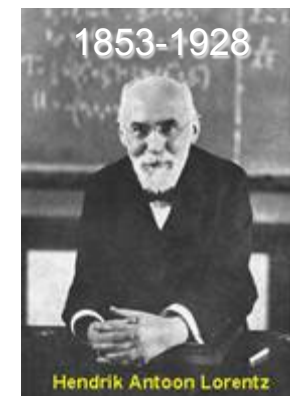
Lorentz Equation

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$W = \int \vec{F} \cdot d\vec{l} = q \int \vec{E} \cdot d\vec{l} + q \int (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

B fields can change the trajectory of a particle
But cannot do *work* and thus change its energy

$$\vec{F} = q\vec{E} \quad W = q \int \vec{E} \cdot d\vec{l}$$



Cyclotron

- In a cyclotron, the charged particles circulate in a strong magnetic field and are accelerated by electric fields in one or more gaps. After having passed a gap, the particles move inside an electrode and are screened from the electric field. When the particles exit from the screened area and enter the next gap, the phase of the time-varying voltage has changed by 180 degrees so that the particles are again accelerated.
- Cyclotron condition:

Centripetal force=Lorentz

$$\text{force} = e[\vec{E} + \vec{v} \times \vec{B}]$$

$$\frac{mv^2}{\rho} = evB$$

$$\rho = \frac{mv}{eB} = \frac{p}{eB}$$

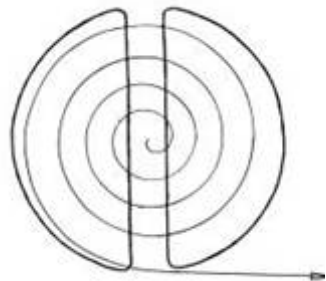
$$f = \frac{v}{2\pi\rho} = \frac{eB}{2\pi m}$$



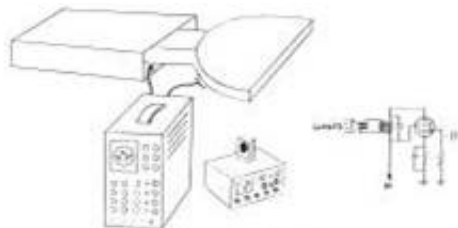
- Only works for non-relativistic particles

The Cyclotron: Different Points of View

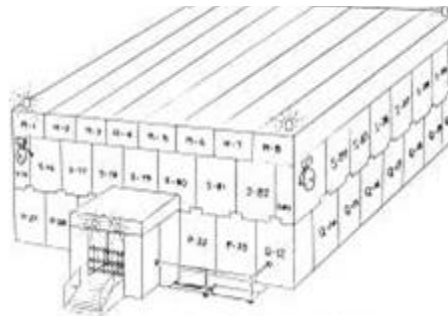
The Cyclotron, as seen by...



... the inventor



... the electrical engineer



... the health physicist

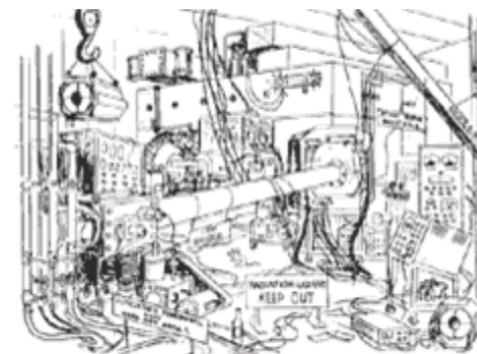


... the experimental physicist

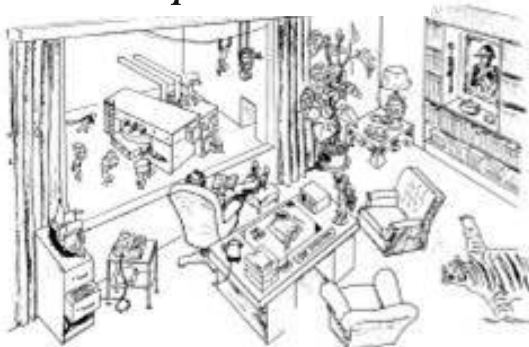


... the operator

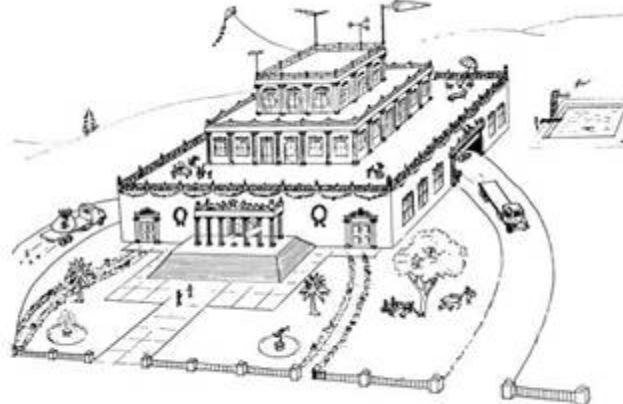
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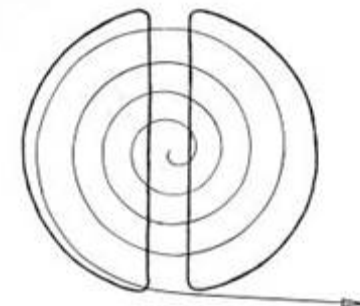
... the visitor



... the laboratory director



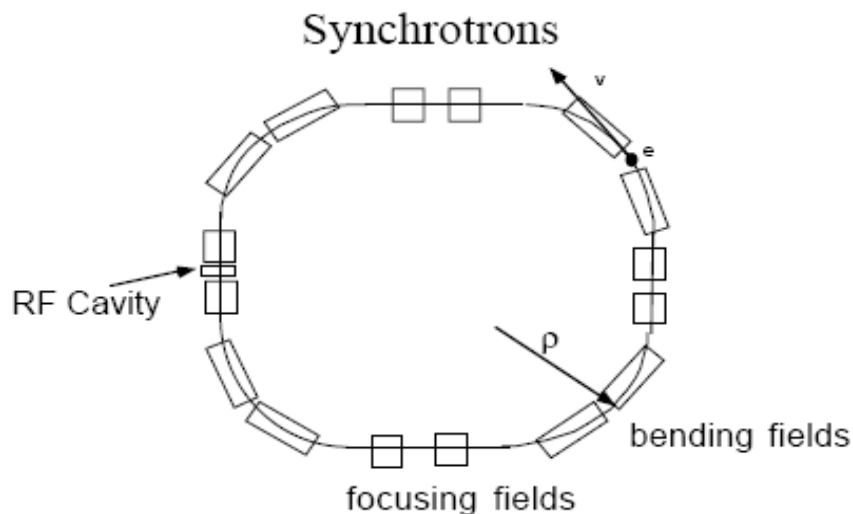
... the governmental funding agency



... the student

Synchrotron (1945)

- Synchrotrons as well as the linear accelerators (linacs) mentioned before, are important in elementary particle physics research, where highest possible particle energies are needed.
- A synchrotron is a circular accelerator which has one (or a few) electromagnetic resonant cavity to accelerate the particles. A constant orbit is maintained during the acceleration.
 - First ones were weak focusing (very large vacuum chambers and magnets)
 - Later strong focusing.
- Originally ramping/cycling, today often storage rings (many h)

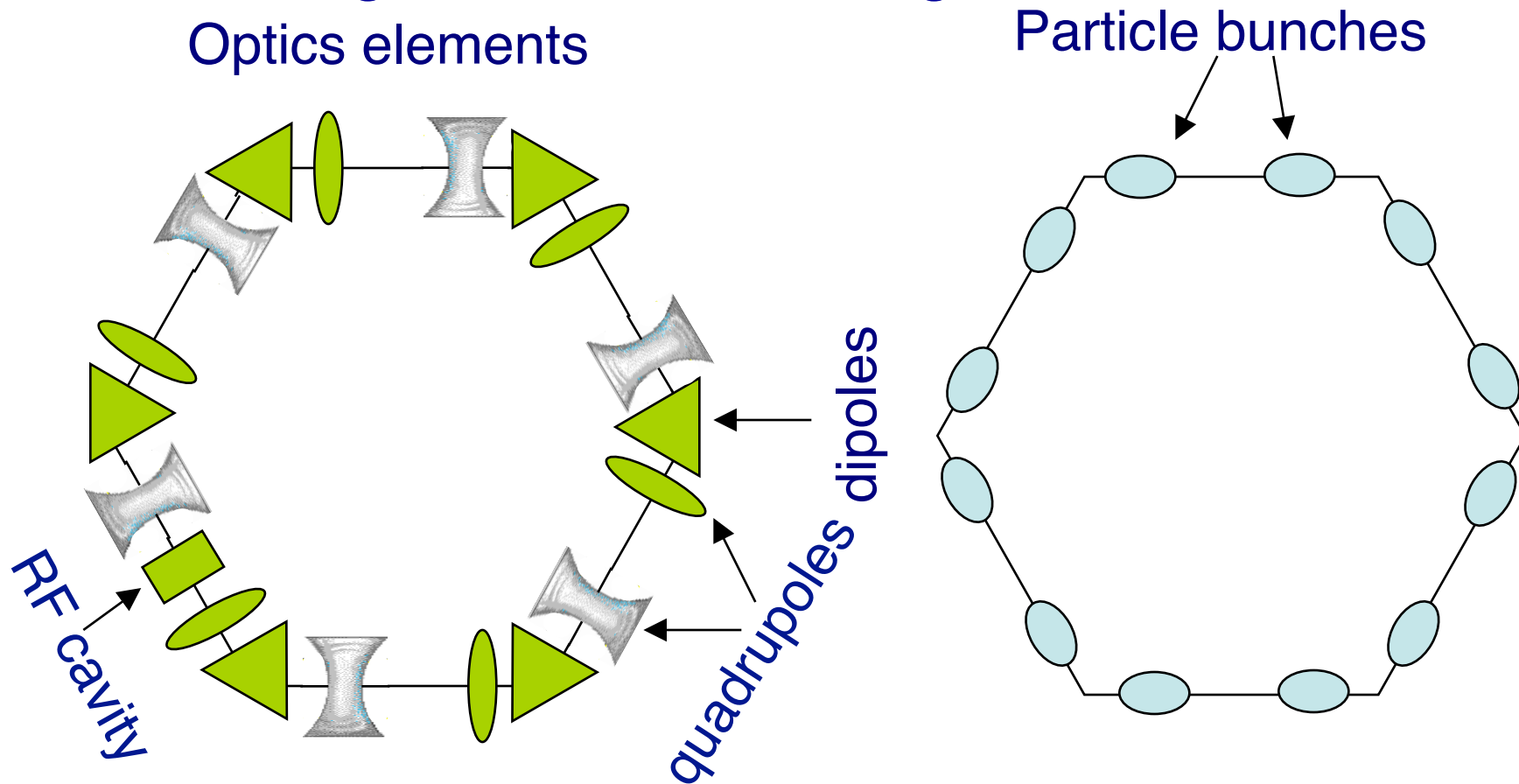


The synchrotron concept seems to have been first proposed in 1943 by the Australian physicist Mark Oliphant.



ALS Synchrotrons / Storage Rings

In a particle storage rings, charged particles circulate around the ring in bunches for a large number of turns.



Equations of Motion in a Storage Ring

The motion of each charged particle is determined by the electric and magnetic forces that it encounters as it orbits the ring:

- Lorentz Force

$$F = ma = e(E + v \times B),$$

m is the relativistic mass of the particle,

e is the charge of the particle,

v is the velocity of the particle,

a is the acceleration of the particle,

E is the electric field and,

B is the magnetic field.

Typical Magnet Types

There are several magnet types that are used in storage rings:

Dipoles → used for guiding

$$B_x = 0$$

$$B_y = B_0$$

Quadrupoles → used for focussing

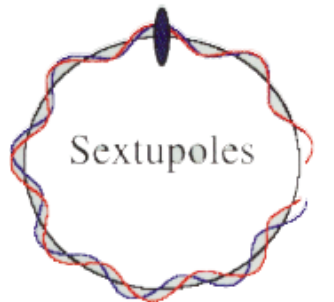
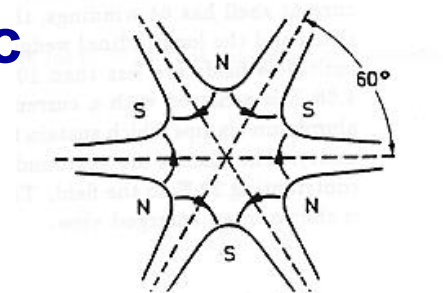
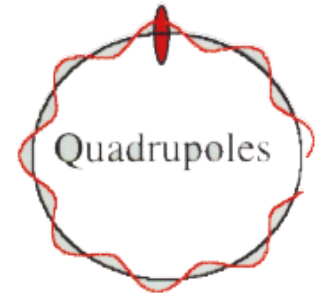
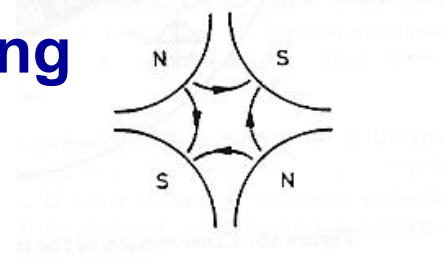
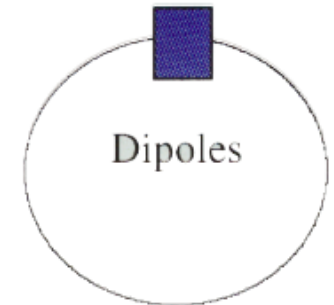
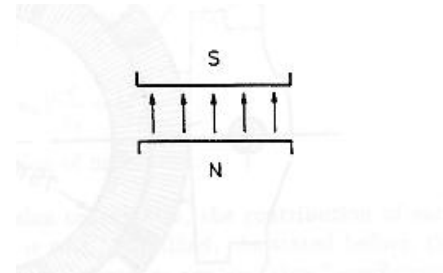
$$B_x = Ky$$

$$B_y = -Kx$$

Sextupoles → used for chromatic correction

$$B_x = 2Sxy$$

$$B_y = S(x^2 - y^2)$$

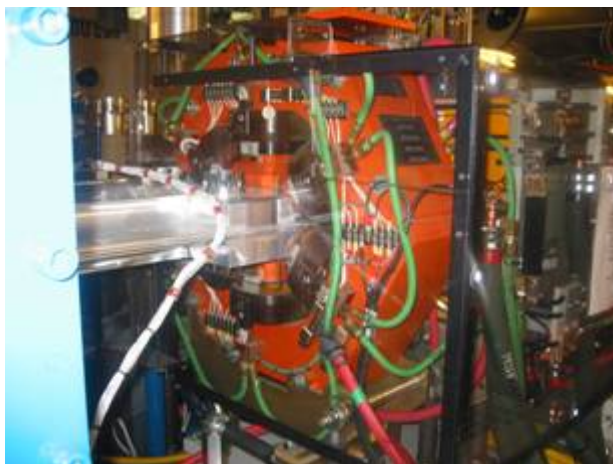


Practical Magnet Examples at the ALS



Quadrupoles

Dipoles



Sextupoles

ALS Differential Equation/Matrix Formalism

There are two approaches to introduce the motion of particles in a storage ring

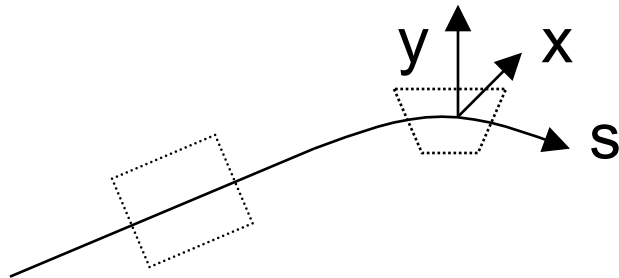
- 1. The traditional way in which one begins with Hill's equation, defines beta functions and dispersion, and how they are generated and propagate, ...**
- 2. The way that our computer models actually do it**

I will begin with the first way (as a brief recap) but spend most of the time with the second approach

Coordinate System

Change dependent variable from time to longitudinal position, s

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



Typically the coordinate system chosen is the one that allows the easiest field representation

ALS First approach – traditional one

This approach (differential equations) provides some insights into concepts but is limited in usefulness for actual calculations

We begin with on-energy no coupling case. The beam is transversely focused by quadrupole magnets. The horizontal linear equation of motion is

$$\frac{d^2 x}{ds^2} = -k(s)x,$$

where $k = \frac{B_T}{(B\rho)a}$, with

B_T being the pole tip field

a the pole-tip radius, and

$$B\rho[\text{T}\cdot\text{m}] \approx 3.356 p[\text{GeV}/c]$$

Hills equation

The solution can be parameterized by a pseudo-harmonic oscillation of the form

$$x_{\beta}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\varphi(s) + \varphi_0)$$

$$x'_{\beta}(s) = -\sqrt{\varepsilon} \frac{\alpha}{\sqrt{\beta(s)}} \cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \sin(\varphi(s) + \varphi_0)$$

where $\beta(s)$ is the beta function,

$\alpha(s)$ is the alpha function,

$\varphi_{x,y}(s)$ is the betatron phase, and

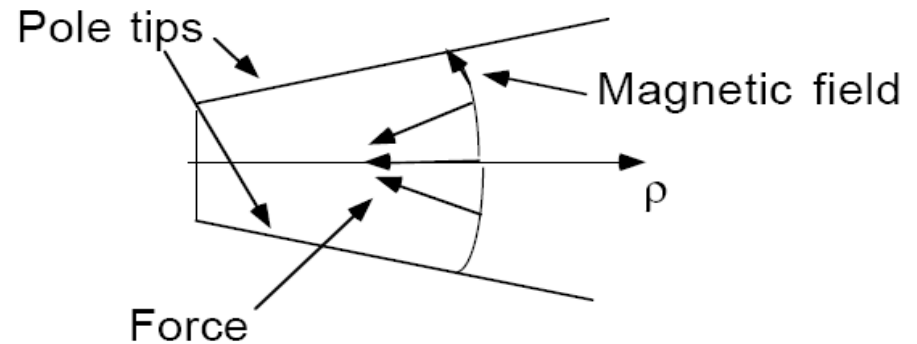
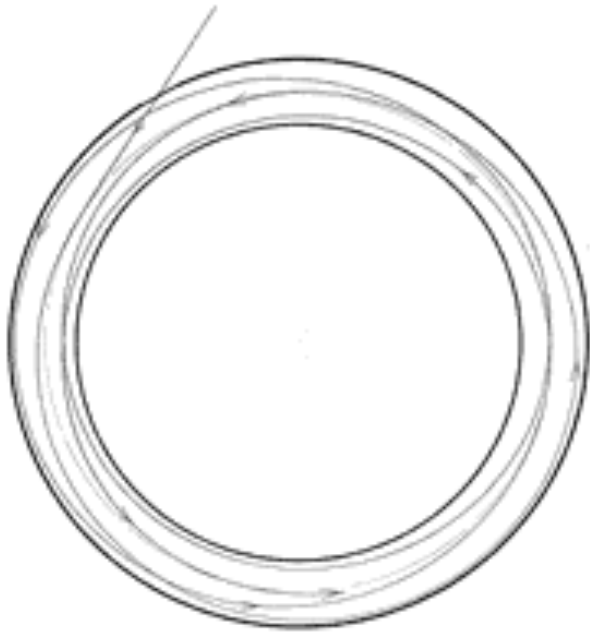
ε is an action variable

$$\varphi = \int_0^s \frac{ds}{\beta}$$

$$\alpha = -\frac{\beta'}{2},$$

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

Weak Focusing



Weak focusing accelerator

- In horizontal plane a homogenous dipole magnetic field 'focuses'
 - Intersecting circles
- Vertically, however, trajectories are diverging for homogenous dipole field.
- Introducing a field gradient provides vertical focusing
 - But also reduces horizontal focusing
 - And causes particles to get out of sync with regards to RF faster

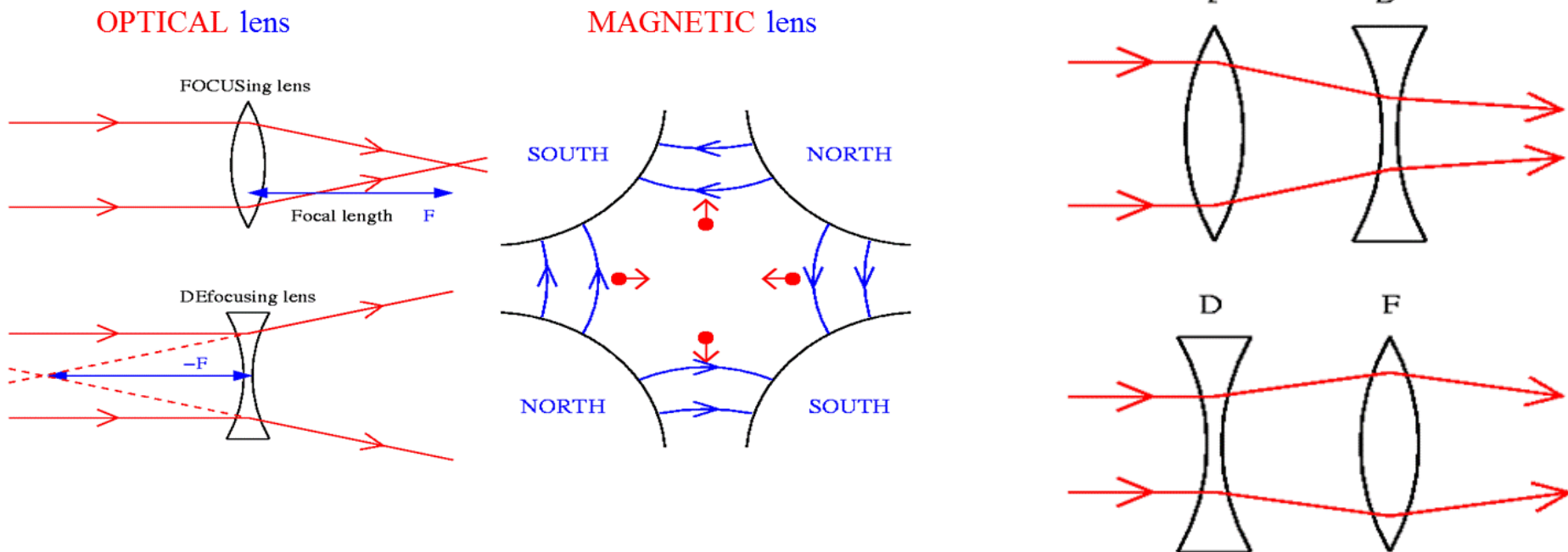
Limitation – Special Relativity

- Relativistic: $\vec{p} = \gamma m \vec{v}$
- Particles in cyclotron become non-synchronous to RF
- Options:
 - Synchrocyclotron, i.e. change RF-frequency over time
 - No cw/multibunch operation possible, i.e. strong reduction in beam current
 - Isochron-cyclotron:
 - Increase field strength radially (causes weak vertical defocusing)
 - Use strong focusing principle instead:
 - See later slides on strong focusing synchrotrons



$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$ is positive for a large range of focal lengths and $d \Rightarrow$ net focusing both radially and vertically

Alternating Gradient Focusing



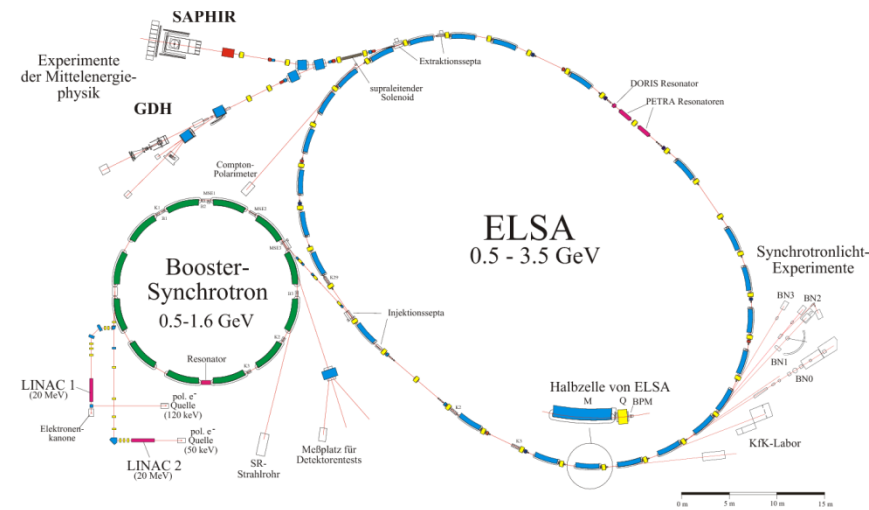
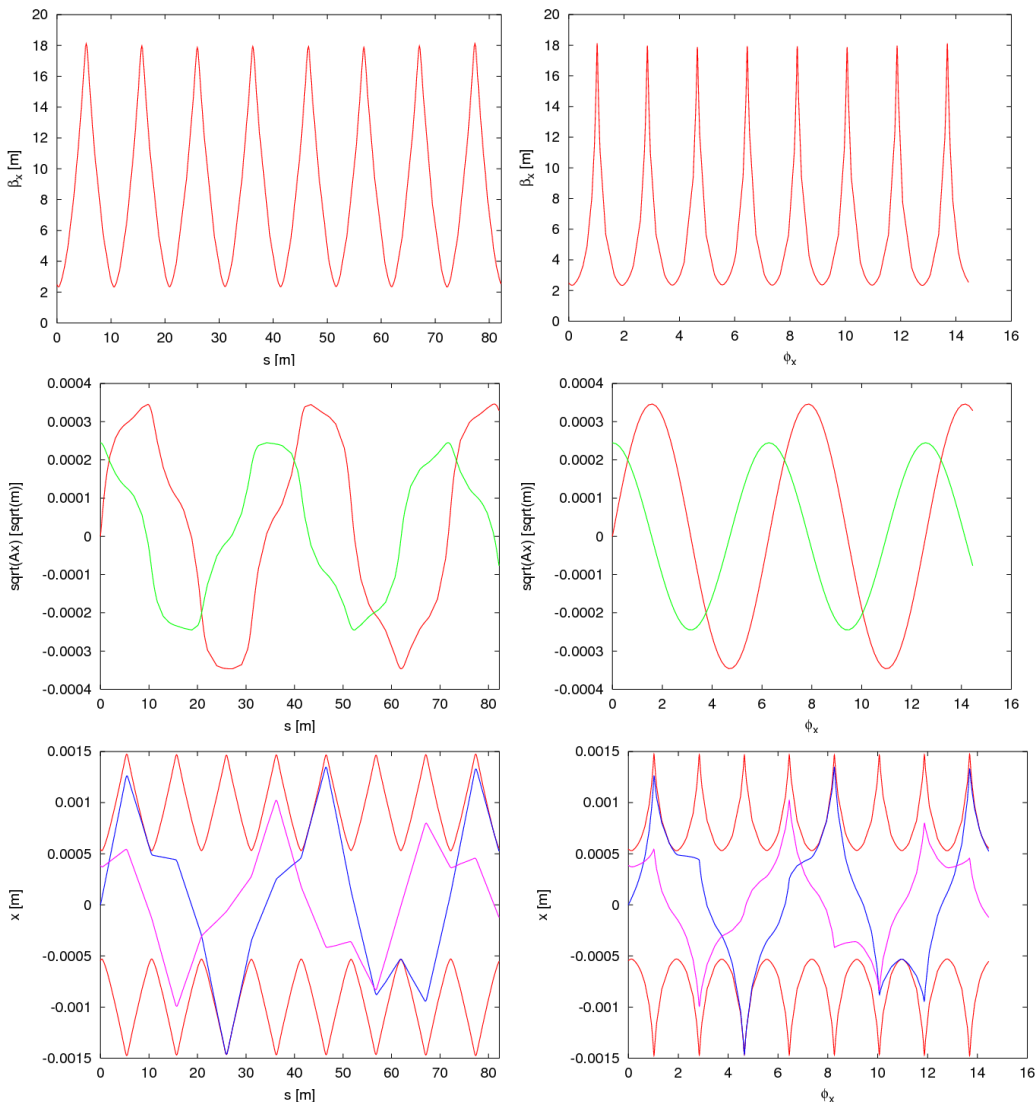
- Magnetic lenses (quadrupoles) cannot be focusing in both planes (Maxwell equations)
- Use alternating gradient / strong focusing
- System of just two quadrupoles can be focusing in both planes

Damped and driven harmonic oscillator – Resonances (will come back Thursday)

- ❖ The general solution is a sum of a transient (the solution for damped undriven harmonic oscillator, homogeneous ODE) that depends on initial conditions, and a steady state (particular solution of the nonhomogenous ODE) that is independent of initial conditions and depends only on driving frequency, driving force, restoring force, damping force, **Damped harmonic oscillator:**

$$\frac{d^2u(t)}{dt^2} + \frac{\omega_0}{Q} \frac{du(t)}{dt} + \omega_0^2 u(t) = \frac{F}{m} \cos(\omega t)$$

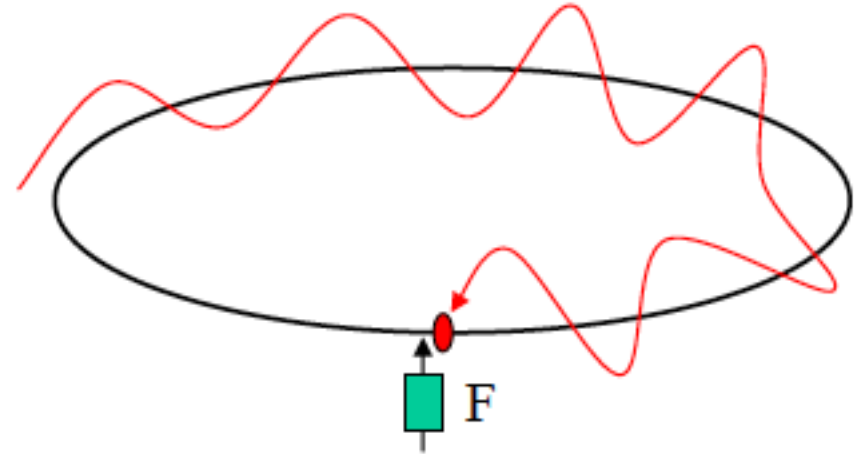
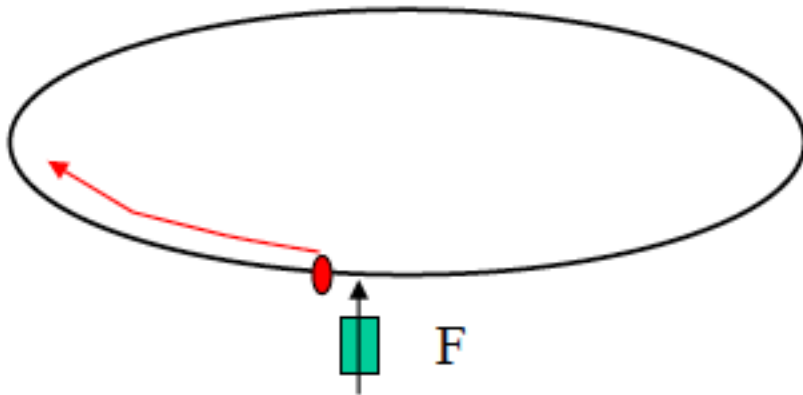
Example of Twiss parameters and trajectories



ELSA (Electron Stretcher and Accelerator) in Bonn is an example of a relatively simple FODO lattice

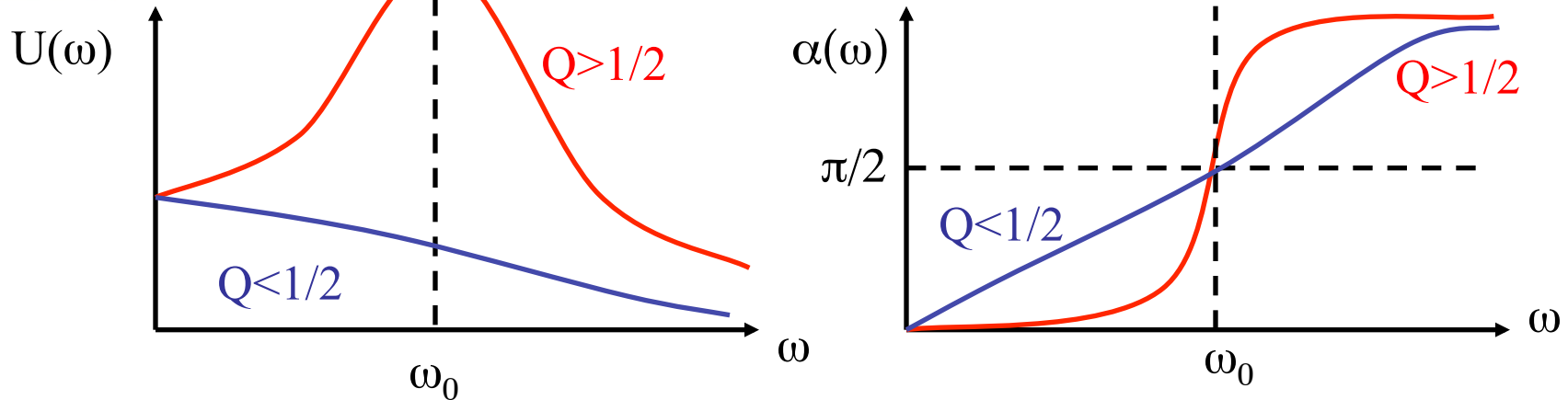
- Beta Function highly periodic
- Trajectories in real space are piecewise straight (with deflections at quadrupoles)
- If one transforms with beta functions and phase advance, they start to look like harmonic functions (sine/cosine)

Driven Oscillator – time dependent excitation vs. fixed in space



- Driven harmonic oscillator
 - periodic excitations
 - frequency of excitation determined by external source
- Betatron oscillations
 - Excitation due to field error, fixed in space (and usually not time dependent)
 - Excitation frequency is determined by oscillation frequency of beam particles
- Both result in similar driven resonances

Resonance effect (more on Wed/Thurs)

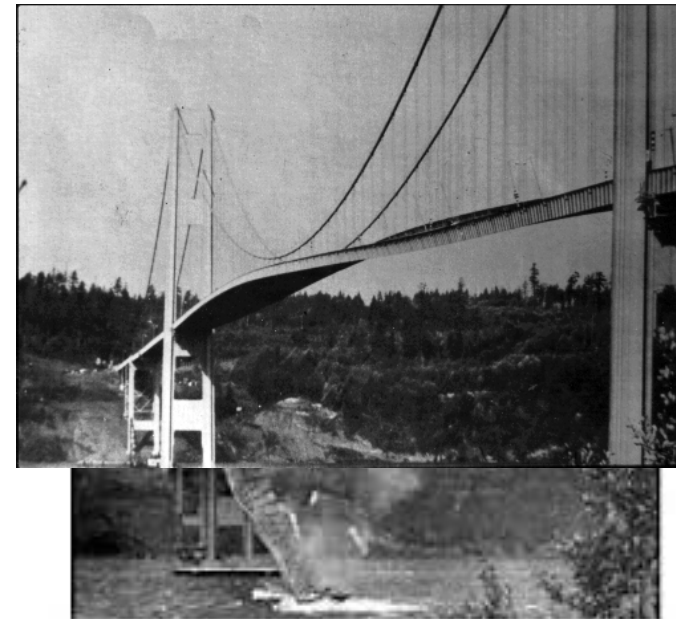


$$U(\omega) = \frac{U(0)}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{Q\omega_0}\right)^2}}$$

- Without or with weak damping a resonance condition occurs for $\omega = \omega_0$
- Infamous example:

Tacoma Narrow bridge 1940

Excitation at the Eigenfrequencies by strong wind



2nd Approach: How to calculate particle trajectories and lattice functions

Begin with equations of motion → Lorentz force

Change dependent variable from time to longitudinal position

Integrate particle trajectory around the ring and find the closed orbit

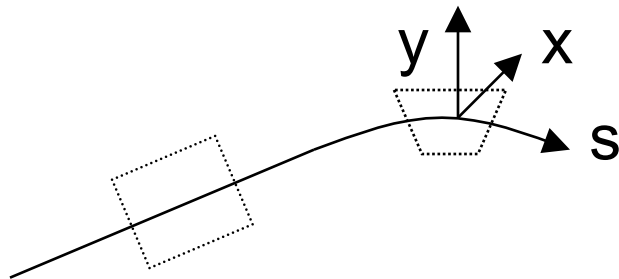
Generate a map around the closed orbit

Analyze and track the map around the ring

Coordinate System

Change dependent variable from time to longitudinal position, s

Coordinate system used to describe the motion is usually locally Cartesian or cylindrical



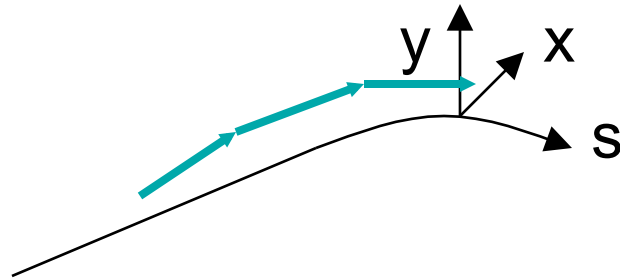
Typically the coordinate system chosen is the one that allows the easiest field representation

Integrate

Integrate through the elements

Use the following coordinates*

$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \quad \tau = \frac{\Delta L}{L}$$

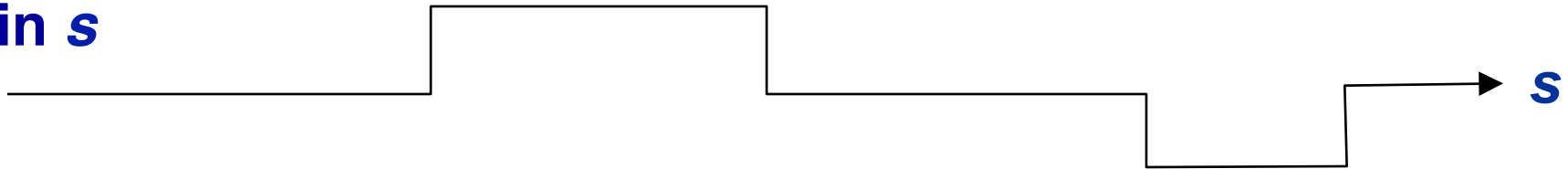


****Note sometimes one uses canonical momentum rather than x' and y'***

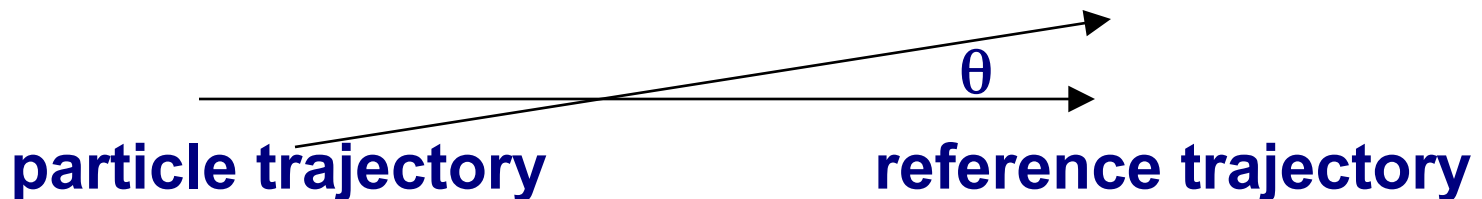
Approximation

Everything up to now there was general. No discussion of the field representation or the integrator. In many codes simplifications are made.

1. The velocity of the particle is the speed of light $\rightarrow v = c$
2. The magnetic field is isomagnetic. Piecewise constant in s

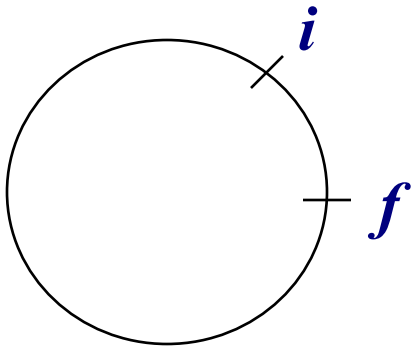


3. The angle of the particles with respect to the reference particle is small and can assume that $\theta = \tan\theta$



Transfer Matrix

One can write the linear transformation between one point in the storage ring (i) to another point (f) as



$$\begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i$$

this is for the case of uncoupled horizontal motion. One can extend this to 4x4 or 6x6 cases.

ALS Piecewise constant magnetic fields

- **General transfer matrix from s_0 to s**

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathcal{M}(s|s_0) \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

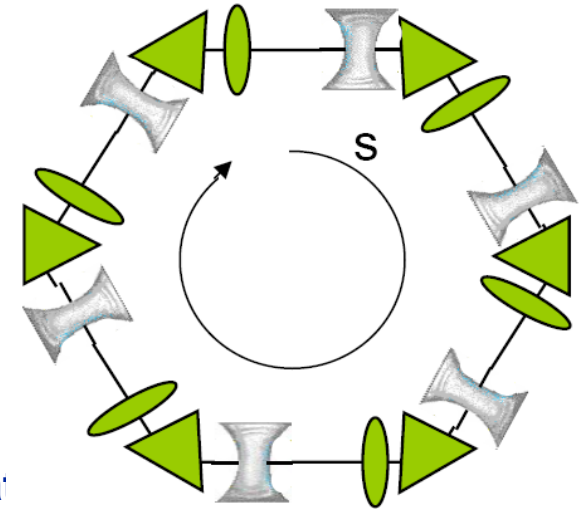
- **Note that**

$$\det(\mathcal{M}(s|s_0)) = C(s|s_0)S'(s|s_0) - S(s|s_0)C'(s|s_0) = 1$$

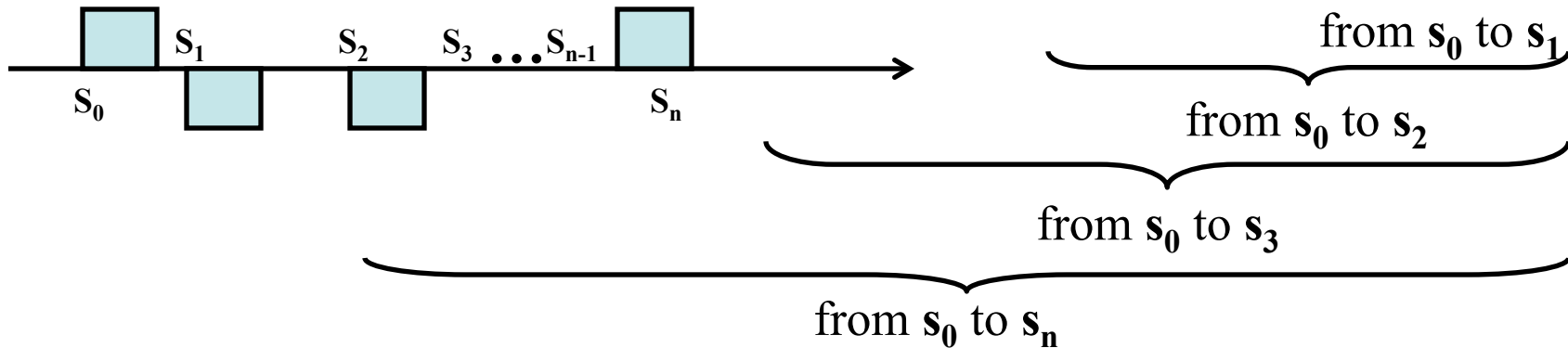
which is always true for conservative systems

- **Note also that** $\mathcal{M}(s_0|s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{I}$

- **The accelerator can be build by a series of matrix multiplica**



$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1}) \dots \mathcal{M}(s_3|s_2) \cdot \mathcal{M}(s_2|s_1) \cdot \mathcal{M}(s_1|s_0)$$



Examples of transfer matrices

Drift of length L

$$R_{drift} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

The matrix for a focusing quadrupole of gradient $k = (\partial B / \partial x) / (B\rho)$ and of length l_q

$$R_{Quad} = \begin{pmatrix} \cos \phi & \sin \phi / \sqrt{|k|} \\ -\sqrt{|k|} \sin \phi & \cos \phi \end{pmatrix}$$

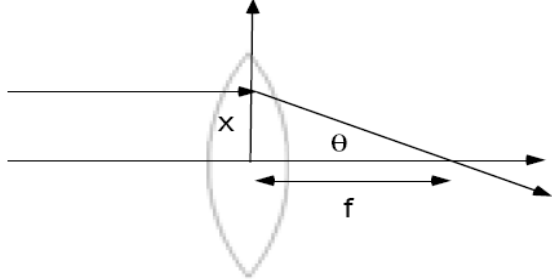
The matrix for a zero length thin quadrupole $K = |k|l_q$

$$R_{thin-lens} = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}$$

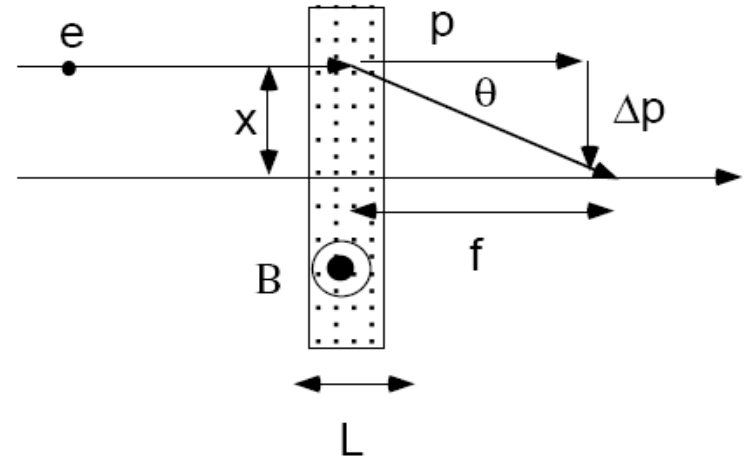
Magnetic lenses: Quadrupoles

Magnetic focusing fields:

Optical analogy: Thin lens, focal length f



$$\theta = \frac{x}{f}$$



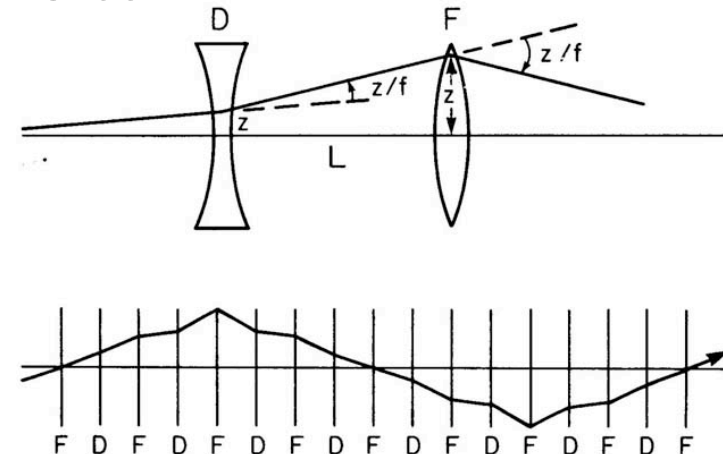
Thin lens representation

$$\begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix}$$

Drift:

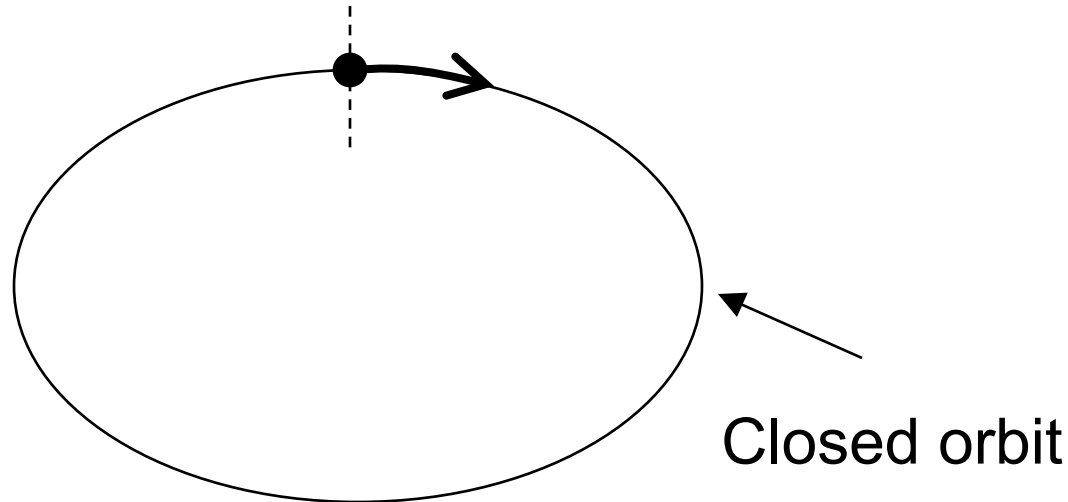
Thin lens:

FODO cell



Find the Closed Orbit

A closed orbit is defined as an orbit on which a particle circulates around the ring arriving with the same position and momentum that it began.



In every working storage ring there exists at least one closed orbit.

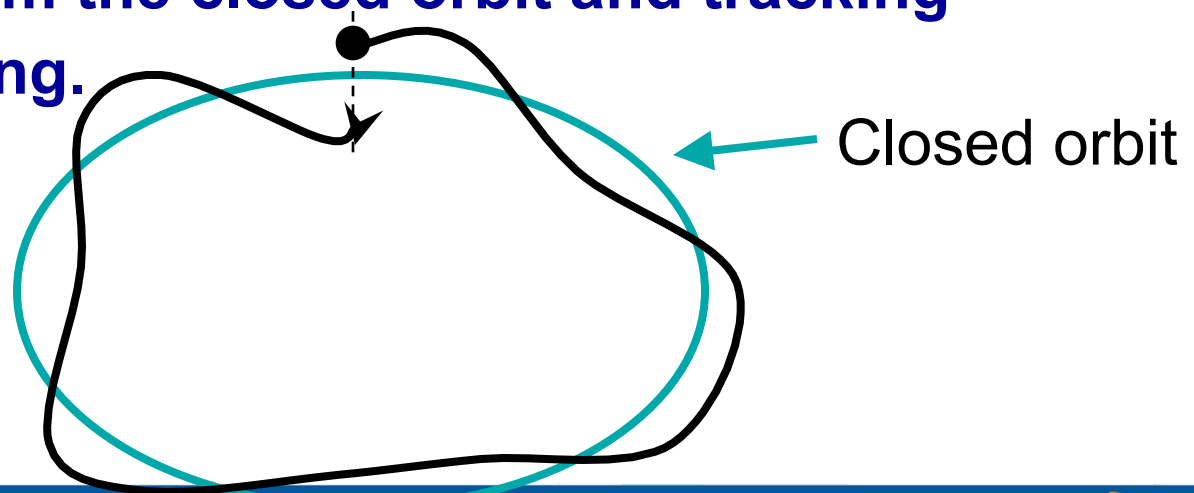
Generate a one-turn Map Around the Closed Orbit

A one-turn map, R , maps a set of initial coordinates of a particle to the final coordinates, one-turn later.

$$x_f = x_i + \frac{dx_f}{dx_i} (x_i - x_{i,co}) + \frac{dx_f}{dx'_i} (x'_i - x'_{i,co}) + \dots$$

$$x'_f = x'_i + \frac{dx'_f}{dx_i} (x_i - x_{i,co}) + \frac{dx'_f}{dx'_i} (x'_i - x'_{i,co}) + \dots$$

The map can be calculated by taking orbits that have a slight deviation from the closed orbit and tracking them around the ring.



Computation of beta-functions and tunes

The one turn matrix (the first order term of the map) can be written

$$R_{\text{one-turn}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos \phi + \alpha \sin \phi & \beta \sin \phi \\ -\gamma \sin \phi & \cos \phi - \alpha \sin \phi \end{pmatrix}$$

Where α, β, γ are called the Twiss parameters

and the betatron tune, $\nu = \phi / (2 * \pi)$

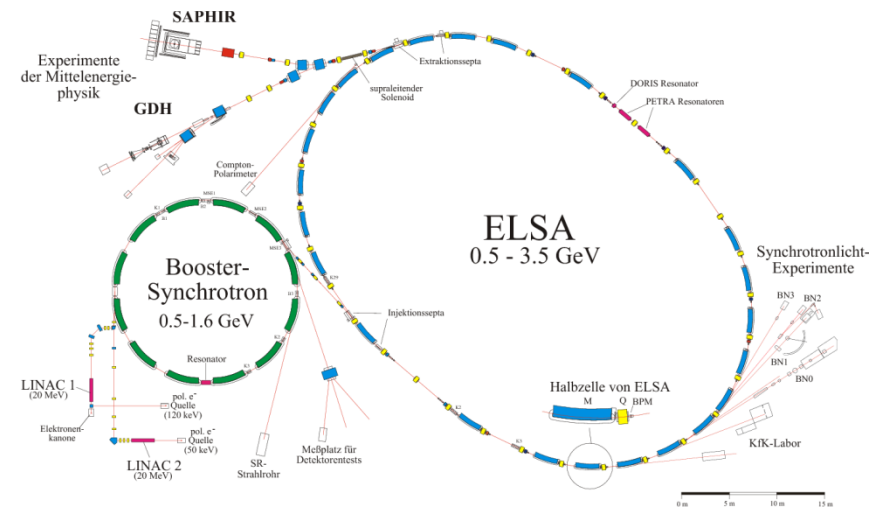
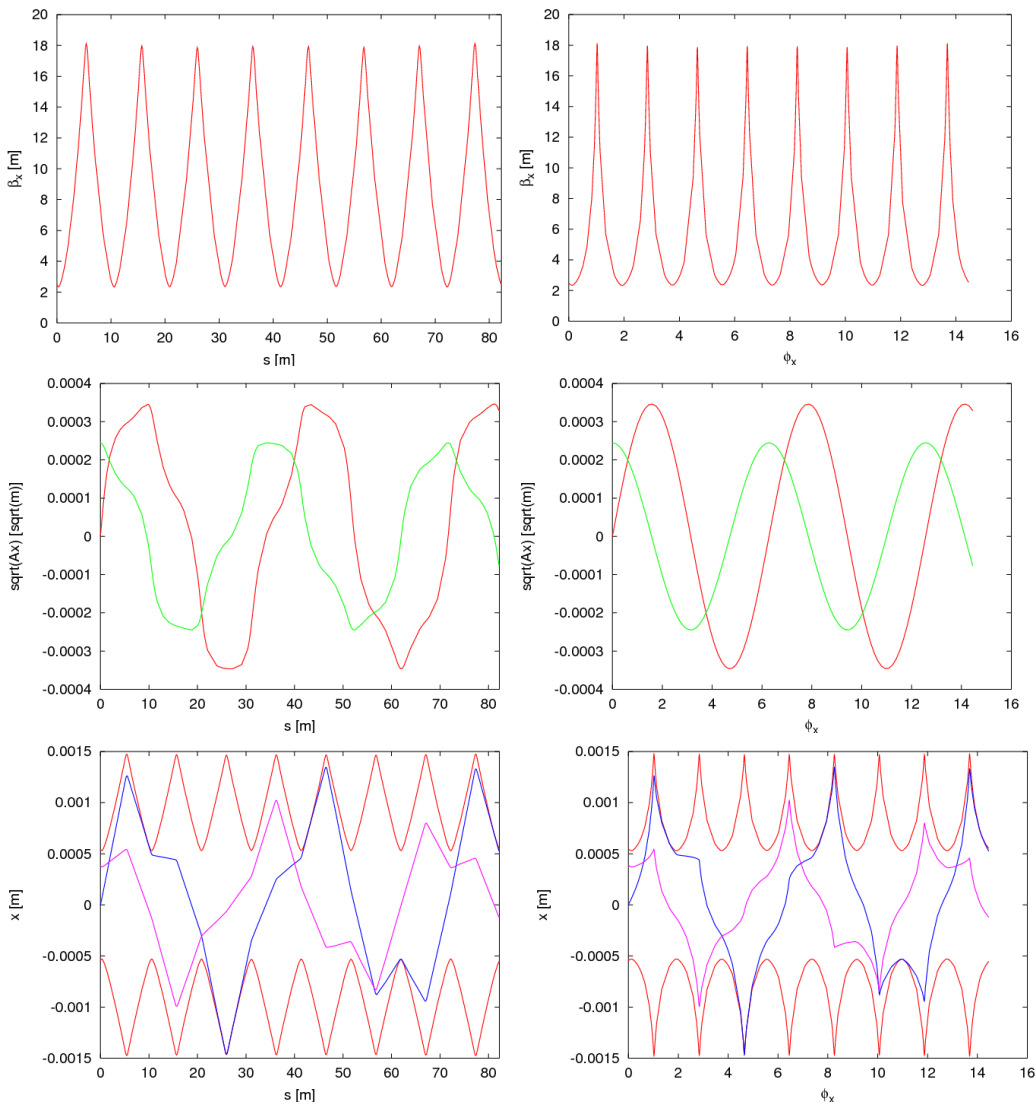
$$\alpha = -\frac{\beta'}{2},$$

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

For long term stability ϕ is real \rightarrow

$$|TR(R)| = |2 \cos \phi| < 2$$

Example of Twiss parameters and trajectories

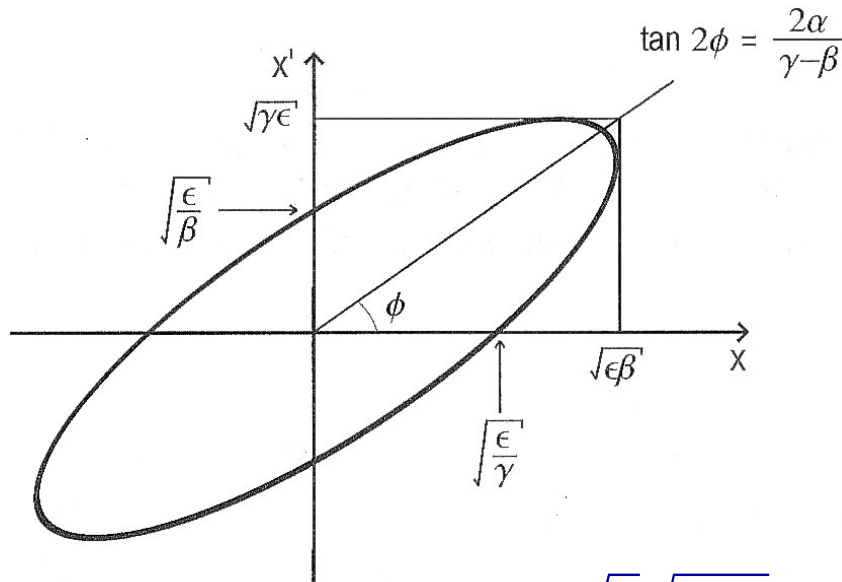


ELSA (Electron Stretcher and Accelerator) in Bonn is an example of a relatively simple FODO lattice

- Beta Function highly periodic
- Trajectories in real space are piecewise straight (with deflections at quadrupoles)
- If one transforms with beta functions and phase advance, they start to look like harmonic functions (sine/cosine)

Beam Ellipse

In an linear uncoupled machine the turn-by-turn positions and angles of the particle motion will lie on an ellipse



Area of the ellipse, ϵ :

$$\epsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

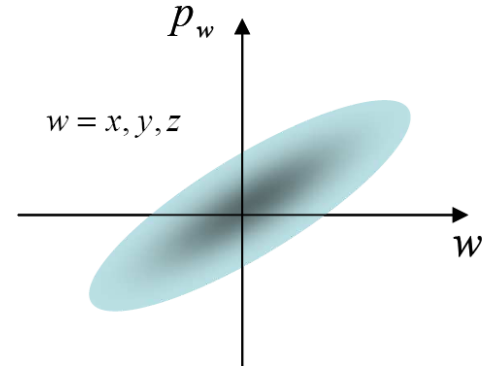
$$x_{\beta}(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\varphi(s) + \varphi_0)$$

$$x'_{\beta}(s) = -\sqrt{\epsilon} \frac{\alpha}{\sqrt{\beta(s)}} \cos(\varphi(s) + \varphi_0) - \frac{\sqrt{\epsilon}}{\sqrt{\beta(s)}} \sin(\varphi(s) + \varphi_0)$$

Emittance Definition

- Consider the decoupled case and use the $\{w, w'\}$ plane where w can be either x or y :
 - The emittance is the phase space area occupied by the system of particles, divided by π

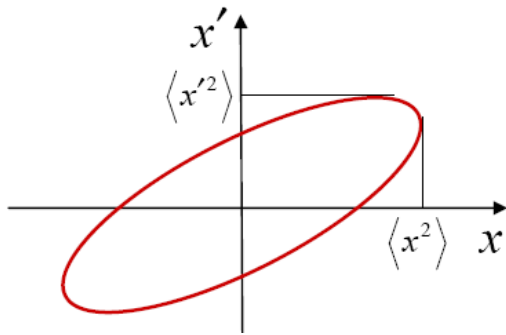
$$\varepsilon_w = \frac{A_{ww'}}{\pi} \quad w = x, y$$



- x' and y' are conjugate to x and y when $B_z = 0$ and in absence of acceleration. In this case, we can immediately apply the Liouville theorem:
 - For such a system the emittance is an invariant of the motion.
- This specific case is very common in accelerators:
 - For most of the elements in a beam transferline, such as dipoles, quadrupoles, sextupoles, ..., the above conditions apply and the emittance is conserved.

Emittance Definition/Statistical

- Emittance defined as the phase space area occupied by an ensemble of particles
- Example: In the transverse coordinates it is the product of the size (cross section) and the divergence of a beam (at beam waists).
- Emittance can be defined as a statistical quantity (beam is composed of finite number of particles)



This is equivalent to associate to the real beam an *equivalent or phase ellipse* in the phase space with area $\pi\epsilon_{rms}$ and equation:

$$\frac{\langle x'^2 \rangle}{\epsilon_{rms}} x^2 + \frac{\langle x^2 \rangle}{\epsilon_{rms}} x'^2 - 2 \frac{\langle x x' \rangle}{\epsilon_{rms}} x x' = \epsilon_{rms}$$

$$\epsilon_{geometric,rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

$$\langle x^2 \rangle = \frac{\sum_{n=1}^N x_n^2}{N} \cong \frac{\int x^2 f_{2D}(x, x') dx dx'}{\int f_{2D}(x, x') dx dx'}$$

ALS Transport of the beam ellipse

Beam ellipse matrix

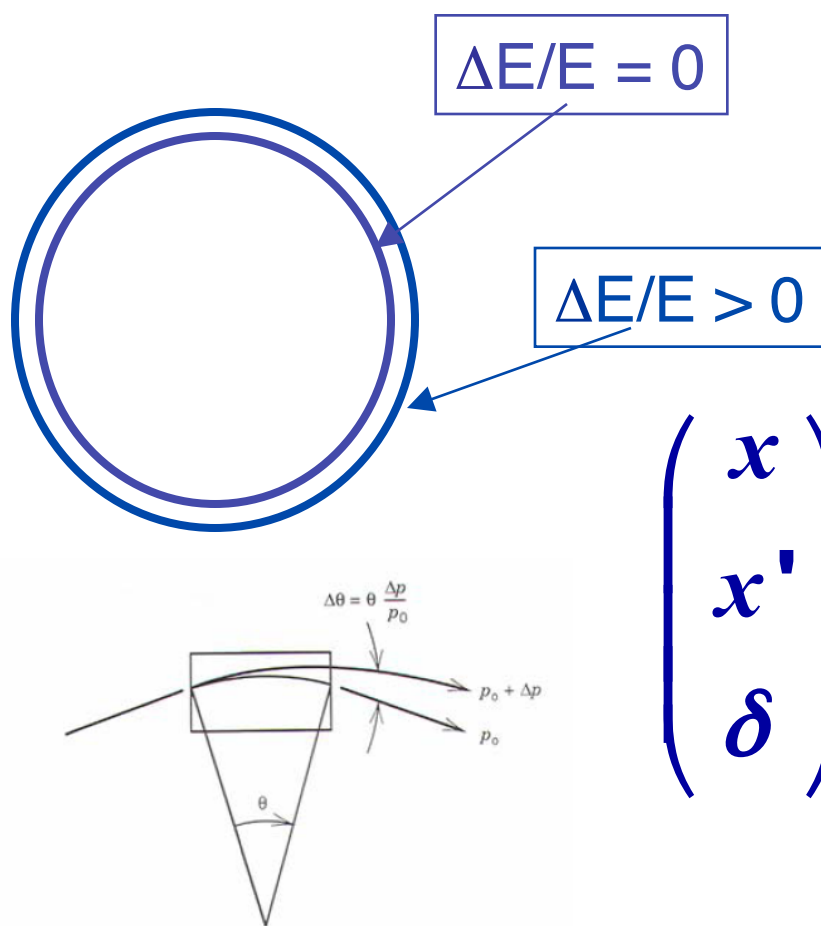
$$\sum_{beam}^x = \varepsilon_x \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transformation of the beam ellipse matrix

$$\sum_{beam,f}^x = R_{x,i-f} \sum_{beam,i}^x R_{x,i-f}^T$$

Off-Energy: Dispersion

Dispersion, D , is the change in closed orbit as a function of energy

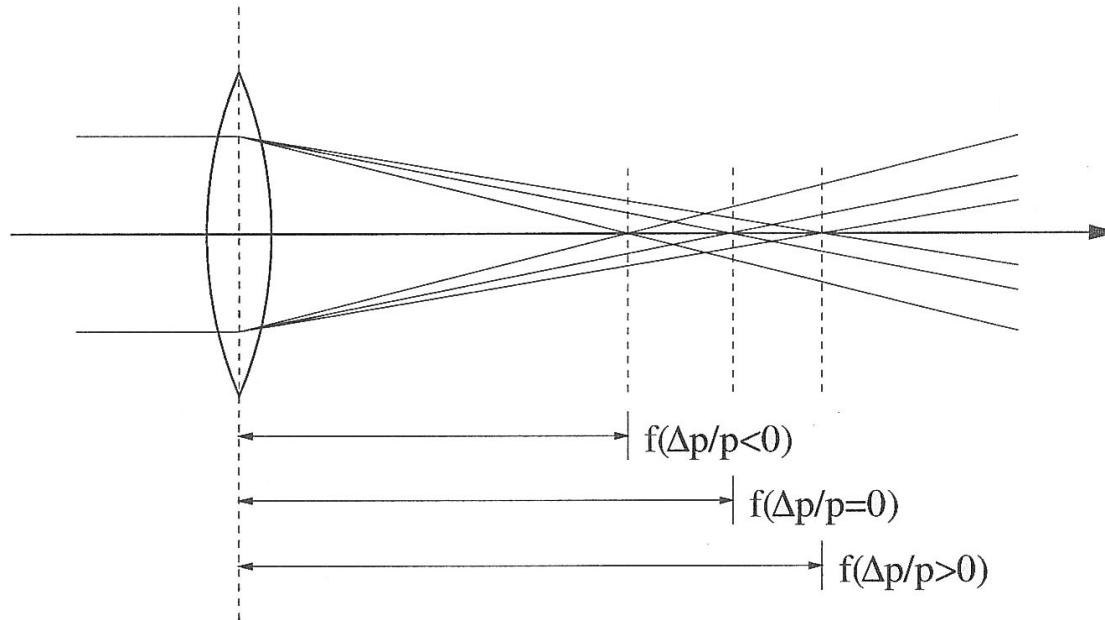


$$\mathbf{x} = D_x \frac{\Delta E}{E}$$

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \delta \end{pmatrix}_f = \begin{pmatrix} C & S & D_x \\ C' & S' & D'_x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{x}' \\ \delta \end{pmatrix}_i$$

What happens off-energy ? Chromatic Aberration

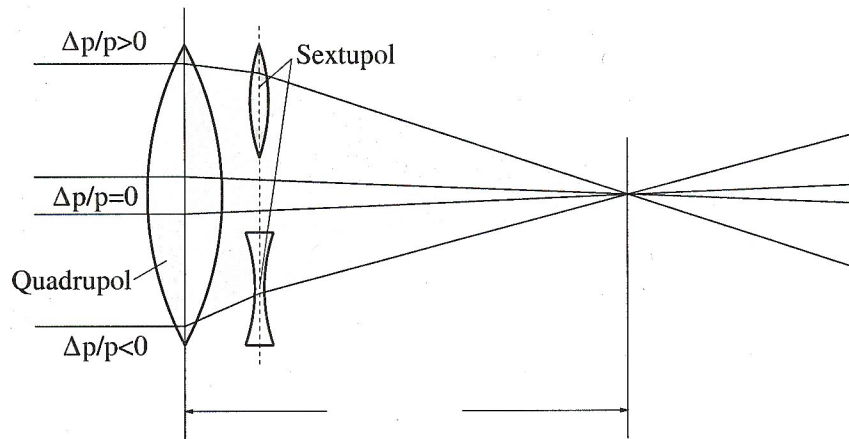
Focal length of the lens is dependent upon energy



Larger energy particles have longer focal lengths

Chromatic Aberration Correction

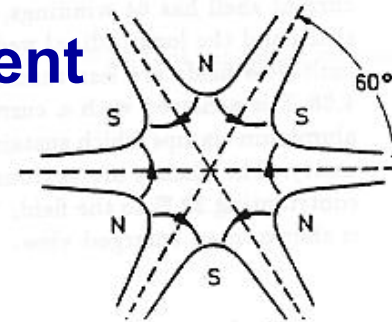
By including dispersion and sextupoles it is possible to compensate (to first order) for chromatic aberrations



The sextupole gives a position dependent Quadrupole

$$B_x = 2Sxy$$

$$B_y = S(x^2 - y^2)$$



Chromatic Aberration Correction

Chromaticity, ν' , is the change in the tune with energy

$$\nu' = \frac{d\nu}{d\delta}$$

Sextupoles can change the chromaticity

$$\Delta\nu_x' = \frac{1}{2\pi} (\Delta S \beta_x D_x)$$

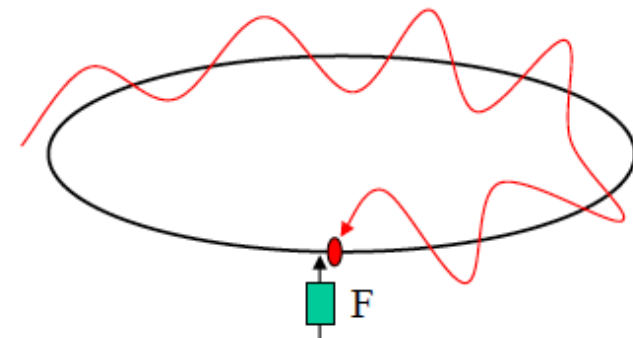
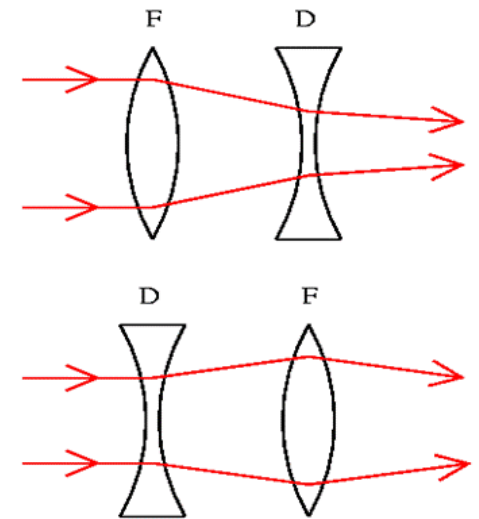
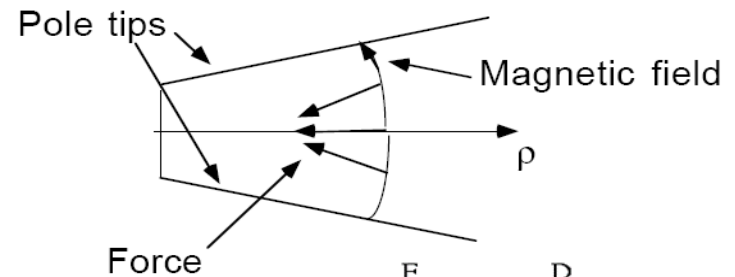
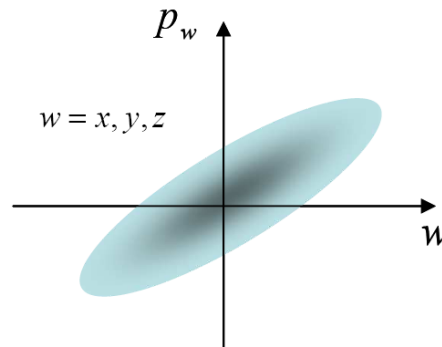
$$\Delta\nu_y' = -\frac{1}{2\pi} (\Delta S \beta_y D_x)$$

where

$$\Delta S = \left(\frac{\partial^2 B_y}{\partial x^2} \right) \text{length} / (2B\rho)$$

- Concepts introduced:
 - Weak Focusing
 - Alternating Gradient Focusing
 - Differential equation treatment (Hill's equation)
 - Resonances (driven harmonic oscillator)
 - Linear Algebra (matrix) treatment
 - Emittance - Liouville

$$\frac{d^2 x}{ds^2} = -k(s)x$$



Longitudinal Dynamics in Storage Rings

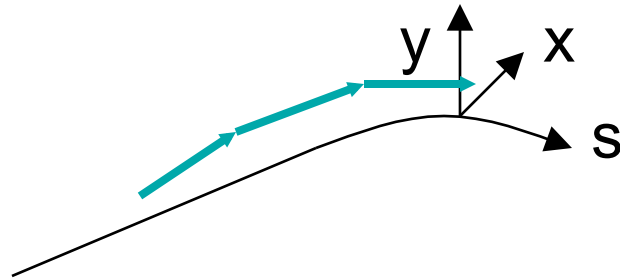
- Let us look at one specific case of acceleration (which has been the most important one for the last 50 years of accelerator physics):
 - RF fields used to accelerate particles
 - In a periodic accelerator (synchrotron or storage ring)
- Similar to what we have already covered in transverse dynamics, the motion becomes (quasi) periodic
 - Will find one important difference: Oscillations are slow compared to revolution period, therefore we do not need beta function formalism
- In addition to velocity term ($1/\gamma^2$), have to take path length into account
 - In general, higher energy particles tend to take wider turns, i.e. they need longer, opposite to the situation at low energies, where higher energy particles are faster

Now to longitudinal Motion: Integrate –

Recap from Before

Integrate through the elements – longitudinally same way as transversely

$$x, \quad x' = \frac{dx}{ds}, \quad y, \quad y' = \frac{dy}{ds}, \quad \delta = \frac{\Delta p}{p_0}, \quad \tau = \frac{\Delta L}{L}$$



Longitudinally:

- **velocity+path length**
- **energy gain/loss (cavities, synchrotron radiation, ...)**

Examples of Element Transfer Matrix

Drift

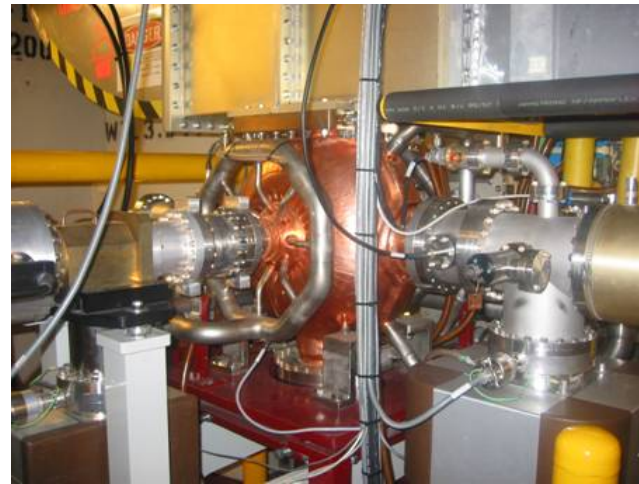
$$\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta^2 \gamma^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

thin RF cavity

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\omega \frac{e\hat{V}}{pc} \cos\phi & 1 \end{pmatrix}$$

coordinate vector

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ ct \\ \frac{\Delta p}{p} \end{pmatrix}$$

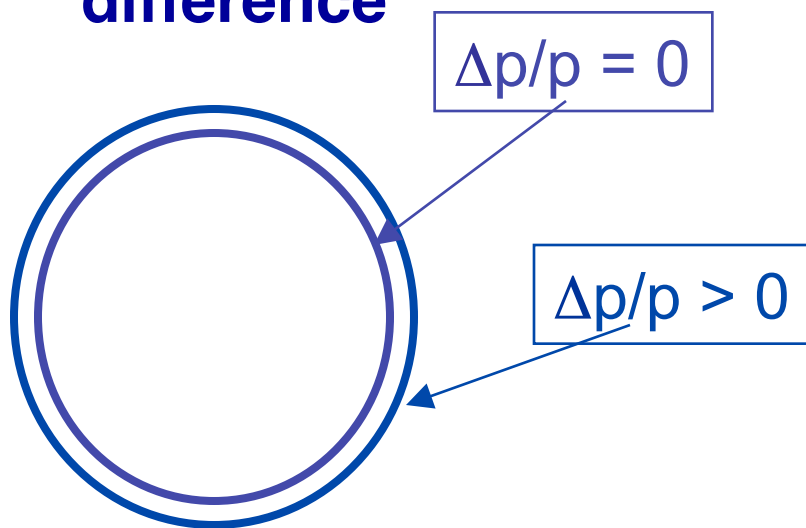


Also: path length effect in dipole –
However, dipole transfer map is pretty confusing, so I will not write it down here ...
(quantitatively, see following slides)

How long does it take to complete revolution?

Assume that the energy is fixed \rightarrow no cavity or damping

- Find the closed orbit for a particle with slightly different energy than the nominal particle. The dispersion is the difference in closed orbit between them normalized by the relative momentum difference

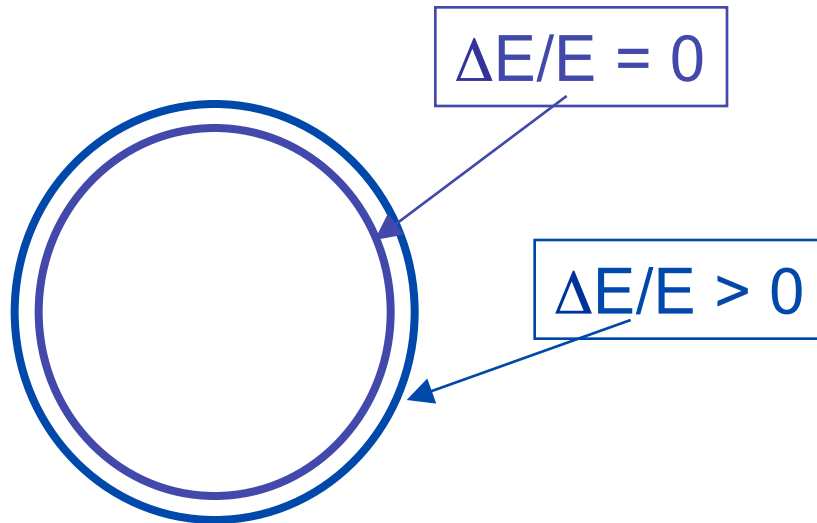


$$x = D_x \frac{\Delta p}{p}, y = D_y \frac{\Delta p}{p}$$

$$x' = D'_x \frac{\Delta p}{p}, y' = D'_y \frac{\Delta p}{p}$$

Momentum Compaction Factor

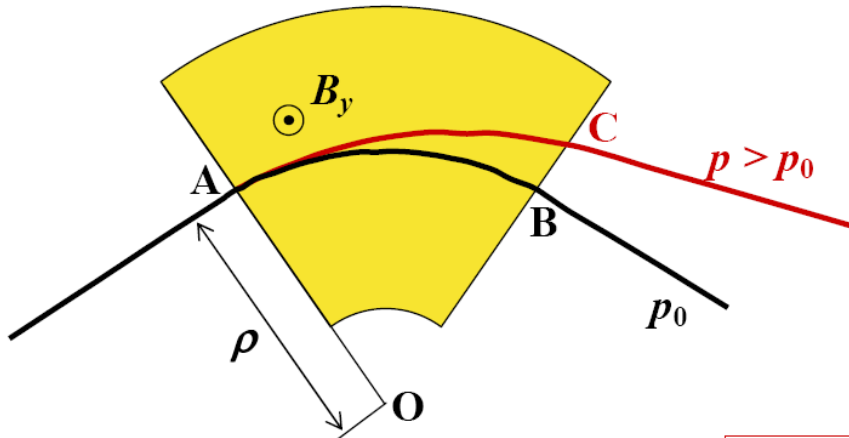
Momentum compaction, α , is the change in the closed orbit length as a function of momentum.



$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

$$\alpha = \int_0^{L_0} \frac{D_x}{\rho} ds$$

Momentum compaction element-by-element



$$\rho = \frac{p}{qB_z} = \frac{\beta \gamma m_0 c}{q B_z}$$

L_0 = Trajectory length between A and B

L = Trajectory length between A and C

$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0}$$



$$\frac{\Delta L}{L_0} = \alpha_C \frac{\Delta p}{p_0}$$

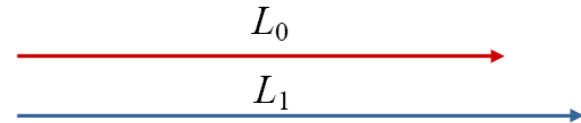
where α_C is constant

- In the example (sector bending magnet) $L > L_0$ so that $\alpha_C > 0$. Higher energy particles will leave the magnet later.

Ballistic time-of-flight

- Consider two particles with different momentum on parallel trajectories:

$$p_1 = p_0 + \Delta p$$



- At a given time t :

$$L_1 = (\beta_0 + \Delta\beta)ct \quad L_0 = \beta_0 ct$$

$$\Rightarrow \frac{\Delta L}{L_0} = \frac{L_1 - L_0}{L_0} = \frac{\Delta\beta}{\beta_0}$$

- But:

$$p = \beta \gamma m_0 c \Rightarrow \Delta p = m_0 c \Delta(\beta \gamma) = m_0 c \gamma^3 \Delta\beta$$

$$\Rightarrow \frac{\Delta p}{p_0} = \gamma^2 \frac{\Delta\beta}{\beta}$$



$$\frac{\Delta L}{L_0} = \frac{1}{\gamma^2} \frac{\Delta p}{p_0}$$

- The ballistic path length dependence on momentum is important everywhere, not just in bending magnets.
- Higher momentum particles are faster, i.e. precede the ones with lower momentum.
- The effect vanishes for relativistic particles.

Phase Slippage, Isochronicity

- Combining the previous two results we obtain the overall phase slippage factor

$$\frac{\Delta s}{L_0} = -\left(\frac{1}{\gamma^2} - \alpha_c\right) \frac{\Delta p}{p_0} = -\eta_c \frac{\Delta p}{p_0}$$

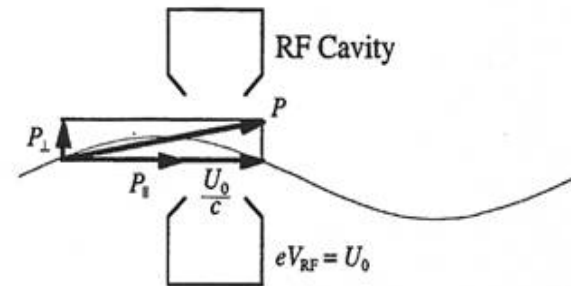
- If $\frac{1}{\gamma^2} = \alpha_c$,

the circulation time does not depend on the particle momentum any more. One calls this isochronous transport

ALS example of RF cavity



- ❖ Cavities replenish the energy loss due to synchrotron radiation



Synchrotron Oscillations

Define the frequency and damping terms:

$$\Omega^2 = \eta_c \left. \frac{1}{p_0} \frac{q}{T_0} \frac{dV}{ds} \right|_{s_0}$$

$$\alpha_D = - \left. \frac{1}{2T_0} \frac{dU}{dE} \right|_{E_0}$$

We obtain the equations of motion for the longitudinal plane:

$$\frac{d^2 \Delta s}{dt^2} + 2\alpha_D \frac{d\Delta s}{dt} + \Omega^2 \Delta s = 0$$

$$\Delta s \ll L_0$$

$$\Delta E \ll E_0$$

$$\Delta E(t) = - \frac{p_0}{\eta_c} \frac{d\Delta s}{dt}$$

ALS Small Amplitude: Damped harmonic oscillator

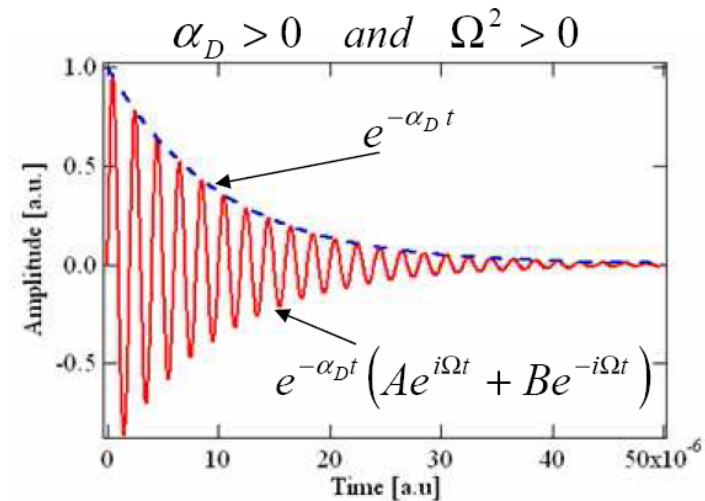
$$\frac{d^2 \Delta s}{dt^2} + 2\alpha_D \frac{d\Delta s}{dt} + \Omega^2 \Delta s = 0$$

This expression is the well known damped harmonic oscillator equation, which has the general solution:

$$\Delta s(t) \cong e^{-\alpha_D t} \left(A e^{i\Omega t} + B e^{-i\Omega t} \right)$$

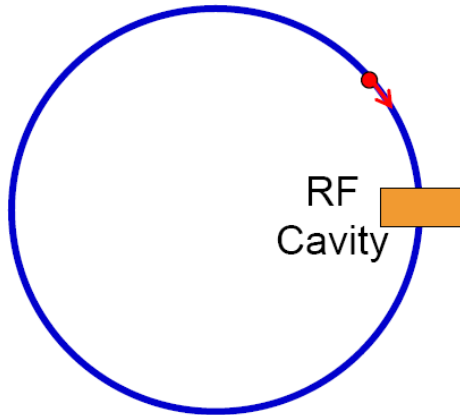
$\alpha_D > 0 \Leftrightarrow$ damped oscillation
 $\alpha_D < 0 \Leftrightarrow$ anti-damped oscillation

$\Omega^2 > 0 \Leftrightarrow$ stable oscillation
 $\Omega^2 < 0 \Leftrightarrow$ unstable motion



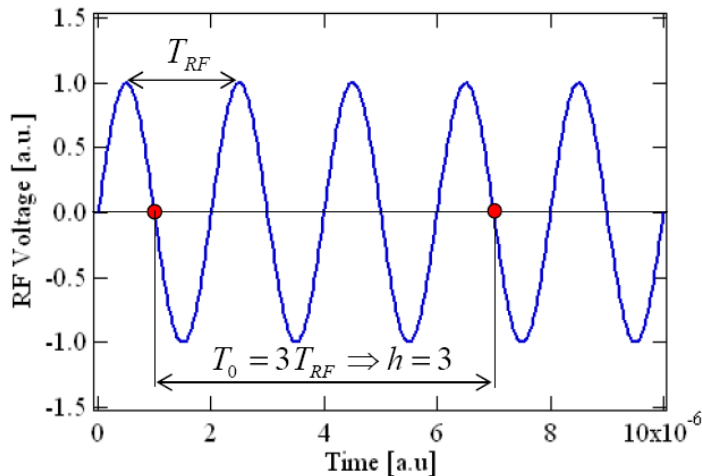
Synchronicity/Harmonic Number

- Let's consider a storage ring with reference trajectory of length L_0 :



$$V_{RF}(t) = \hat{V} \sin(\omega_{RF} t)$$

$$T_0 = \frac{L_0}{\beta c} \quad T_{RF} = \frac{1}{f_{RF}} = \frac{2\pi}{\omega_{RF}}$$



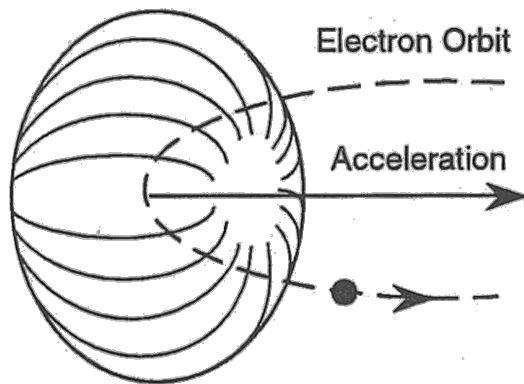
$$T_0 = h T_{RF} \Rightarrow f_0 = \frac{f_{RF}}{h}$$

Synchronicity Condition

The integer h is called the *harmonic number*

Synchrotron Radiation

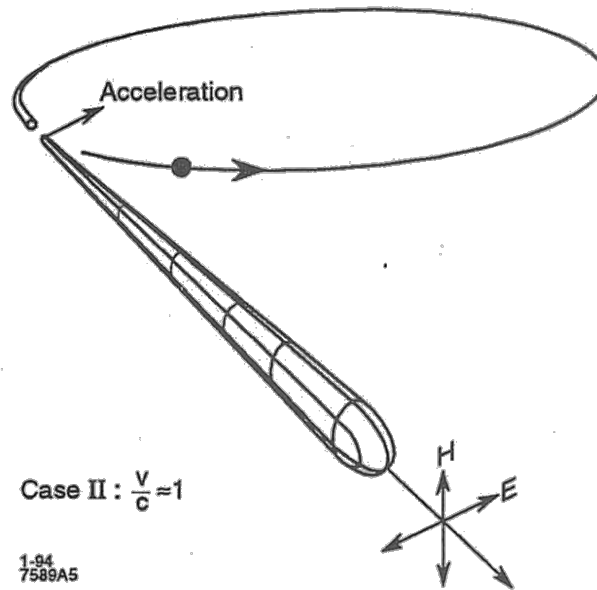
- Radiated power increases at higher velocities
- Radiation becomes more focused at higher velocities



Case I: $\frac{v}{c} \ll 1$

1-94
7589A4

At low electron velocity (non-relativistic case) the radiation is emitted in a non-directional pattern

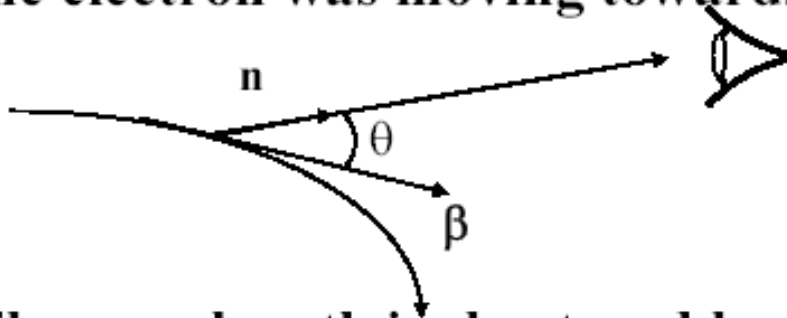


1-94
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When the electron velocity approaches the velocity of light, the emission pattern is folded sharply forward. Also **the radiated power goes up dramatically**

Time compression

Electron with velocity β emits a wave with period T_{emit} while the observer sees a different period T_{obs} because the electron was moving towards the observer



$$T_{obs} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) T_{emit}$$

The wavelength is shortened by the same factor

$$\lambda_{obs} = (1 - \beta \cos \theta) \lambda_{emit}$$

in ultra-relativistic case, looking along a tangent to the trajectory

$$\lambda_{obs} = \frac{1}{2\gamma^2} \lambda_{emit}$$

since

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \approx \frac{1}{2\gamma^2}$$

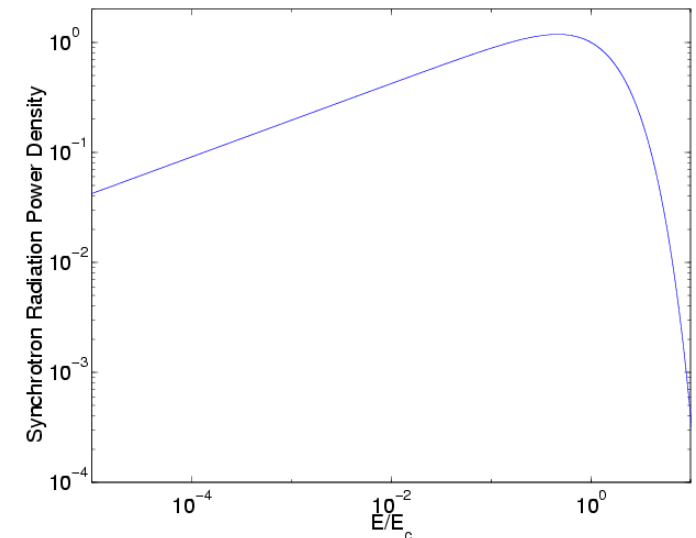
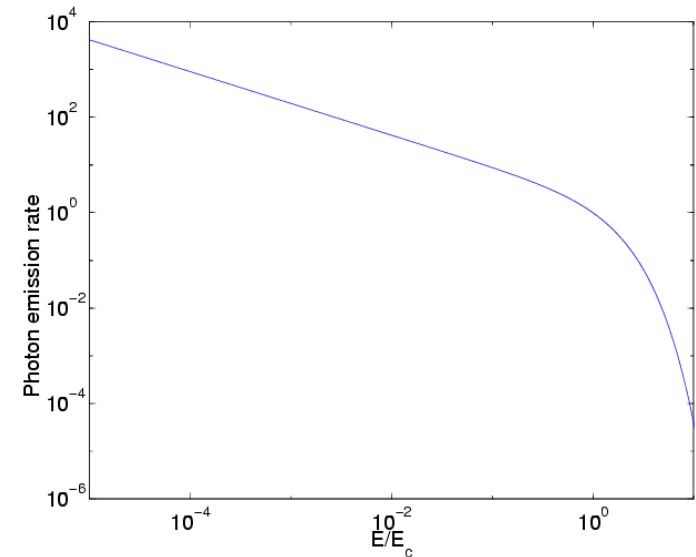
Radiation

The power emitted by a particle is

$$P_{SR} = \frac{2}{3} \alpha hc^2 \frac{\gamma^4}{\rho^2}$$

and the energy loss in one turn is

$$U_0 = \frac{4\pi}{3} \alpha hc \frac{\gamma^4}{\rho^2}$$



Radiation damping

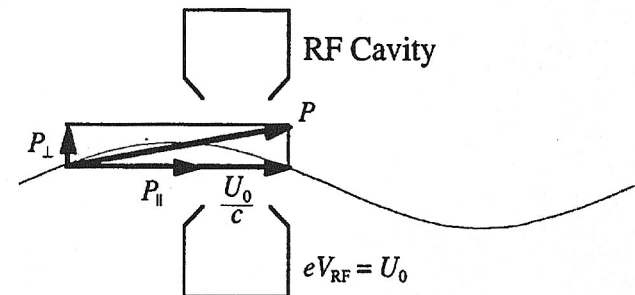
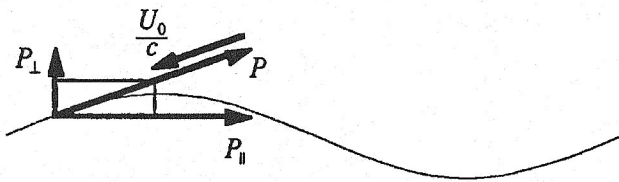
Energy damping:

Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3} \alpha h c^2 \frac{\gamma^4}{\rho^2}$$

Transverse damping:

Energy loss is in the direction of motion while the restoration in the s direction



ALS Quantum excitation - Longitudinally

The synchrotron radiation emitted as photons, the typical photon energy is

$$u_c = h\omega_c = \frac{3}{2} hc \frac{\gamma^3}{\rho}$$

The number of photons emitted is

$$N = \frac{4}{9} \alpha c \frac{\gamma}{\rho}$$

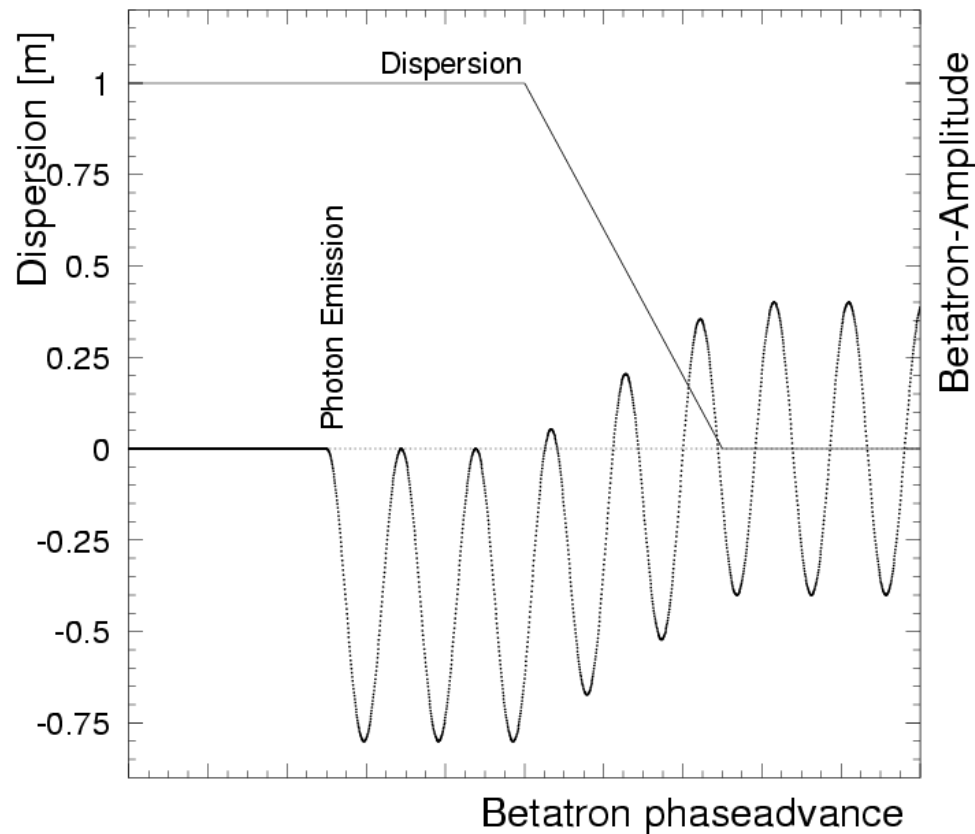
With a statistical uncertainty of \sqrt{N}

The equilibrium energy spread and bunch length is

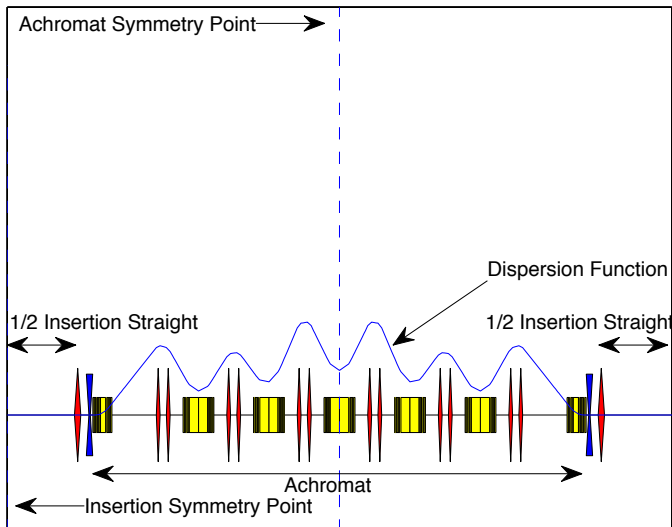
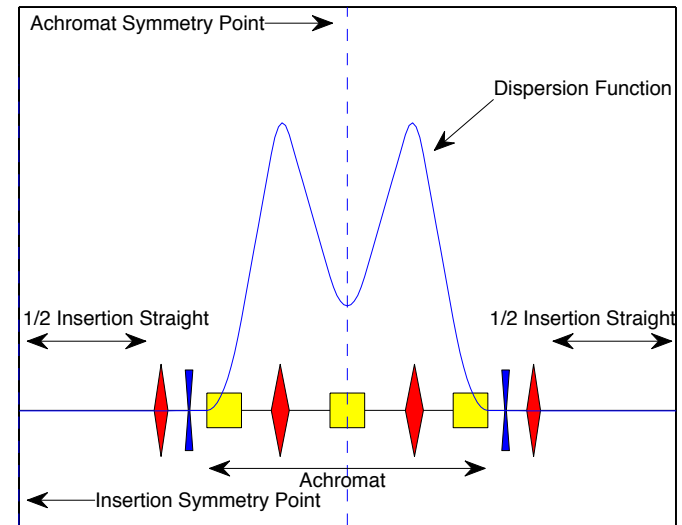
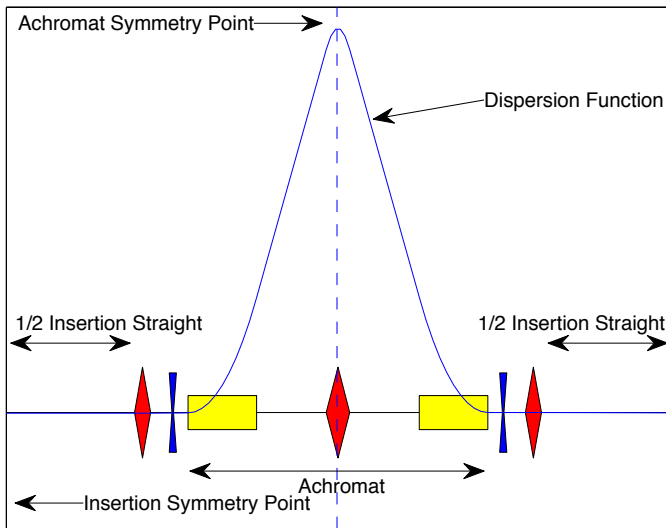
$$\left(\frac{\sigma_e}{E} \right)^2 = 1.468 \cdot 10^{-6} \frac{E^2}{J_\varepsilon \rho} \quad \text{and} \quad \sigma_L = \frac{\alpha R}{f_0} \sigma_e$$

Quantum Excitation - Transversely

Particles change their energy in a region of dispersion undergoes increase transverse oscillations. This balanced by damping gives the equilibrium emittances.



Example: Low Emittance Lattices



- Early 3rd generation SR sources all used double/triple bend achromats (some with gradient dipoles)
- Later optimization included detuning from achromatic condition (Optimizing effective emittance)
- New designs (including USRs) employ MBA
- Damping wigglers can help (emittance, damping time, IBS) but trade energy spread

Time Scales for Particle Dynamics in Rings

- At this point we have discussed the motion of a particle in an accelerator for all 6 phase space dimensions (4 transverse dimensions and 2 longitudinal ones)
- An important effect is that the time scales for different phenomena are quite different:
 - Damping: several ms for electrons, \sim infinity for heavier particles
 - Betatron oscillations: \sim tens of ns
 - Synchrotron oscillations: \sim tens of μ s
 - Revolution period: \sim hundreds of ns to μ s

Summary Ring Long. Dynamics

- Velocity variation often negligible (relativistic)
- Instead Path length difference very important
- Matrix treatment similar to transverse case
 - Can combine both
- Periodic solution (closed orbit)
 - Downstream affects Upstream ...

Summary

- Recapped many concepts of linear beam dynamics
- Matrix (transverse) beam transport approach is helpful to calculate simple problems by hand
- Computer codes use an extension of this approach (nonlinear integrators for individual elements, symplectic, ...)
- Find closed orbit – generate map around closed orbit – lattice functions
- Historic and text book approach of Hill's equation is not very useful for practical purposes (either calculations by hand or computer codes)

List of Literature/Text Books

- Particle Accelerator Physics I (2nd edition, 1998), by Helmut Wiedemann, Springer
 - Or at a more advanced level: Particle Accelerator Physics II, H. Wiedemann, Springer (nonlinear dynamics, etc.)
- D.A. Edwards and M.J. Syphers, An Introduction to the Physics of High Energy Accelerators, John Wiley & Sons (1993)
- Accelerator Physics, S.Y. Lee, World Scientific, Singapore, 1999 (ISBN 9810237103)
- Many nice proceedings of CERN accelerator schools can be found at http://cas.web.cern.ch/cas/CAS_Proceedings.html , for the purpose of this class especially
 - CERN 94-01 v1 + v2
 - CERN 95-06 v1 + v2 (Advanced Class)
 - CERN 98-04 (Synchrotron Radiation+Free Electron Lasers)
- Accelerators and Nobel Laureates” by Sven Kullander which can be viewed at: http://nobelprize.org/nobel_prizes/physics/articles/kullander/

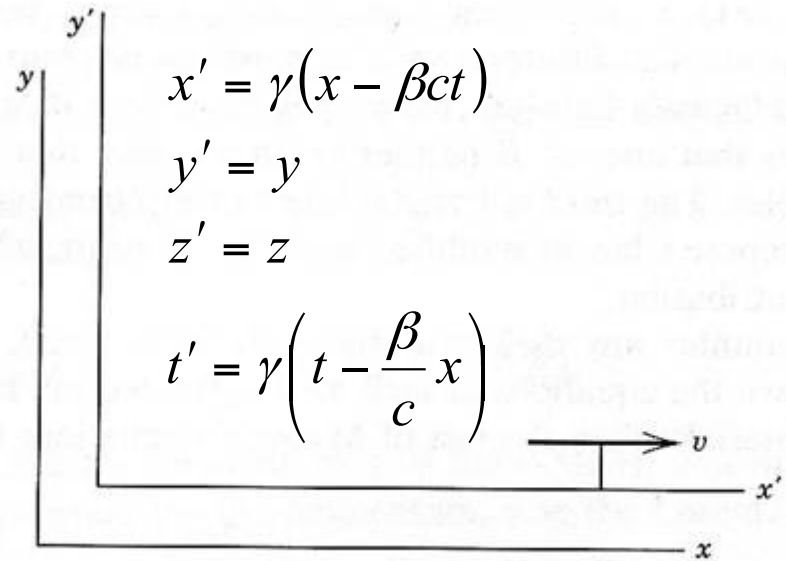
Backup Slides

Special Relativity

- The Lorentz Transformation

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \frac{E_T}{m_0 c^2}$$



- The electron case:

$$m_0 c^2 \cong 0.511 \text{ MeV}$$

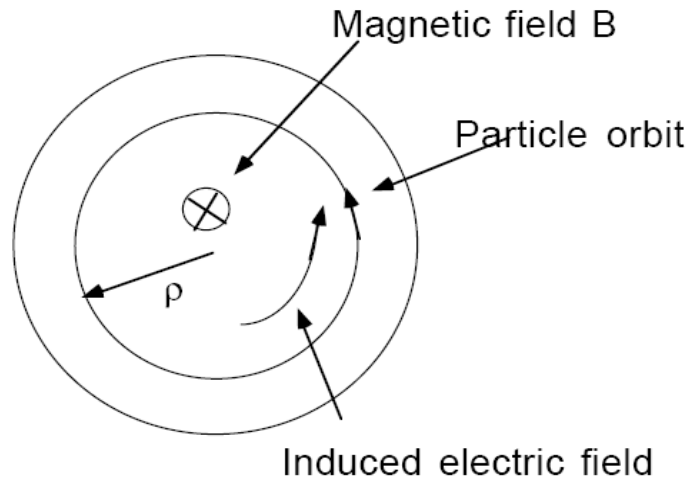
$$\text{If } E_T = 10 \text{ MeV} \Leftrightarrow \gamma \cong 20.6 \Leftrightarrow \beta \cong 0.9988 \approx 1$$

For energies of ~ 10 MeV and above, electrons are “relativistic” particles traveling at the speed of light.

ALS Induction Accelerators

$$\nabla \times \vec{E} = - \frac{d\vec{B}}{dt}$$

Faraday's law



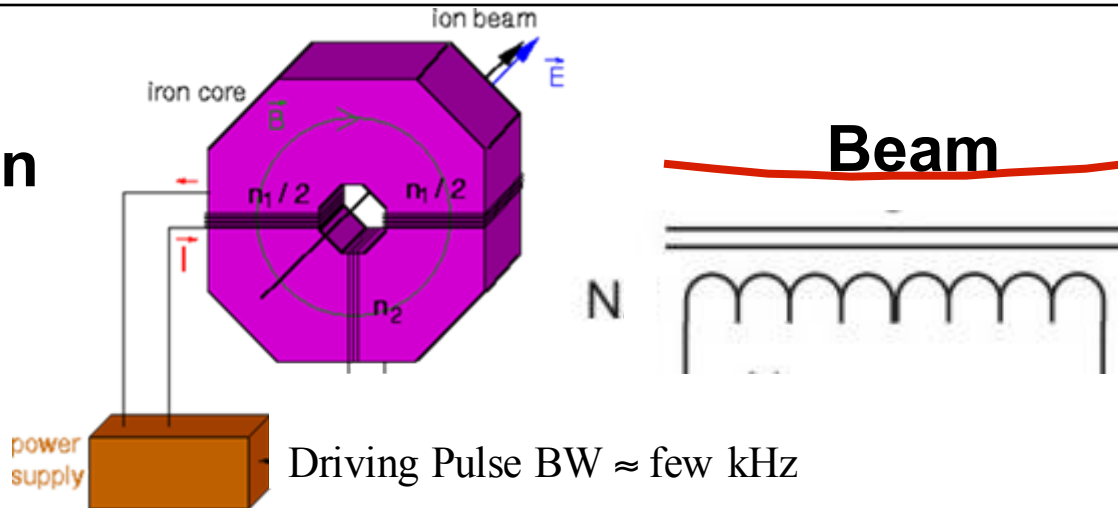
Betatron (Kerst 1940)

- For electrons (reach relativistic velocities quickly) one can use pulsed machines
- Acceleration provided by induced voltage due to change in magnetic field
- Geometry determines synchronicity

$$\Delta\phi = 2\pi\rho^2 B_{\max}$$

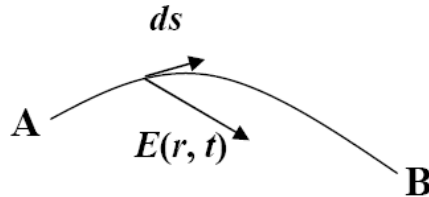
Induction accelerators can be very efficient (>50%) and allow for very high currents (~ 1kA) at relatively moderate energies (few MeV)

Induction Linac (1964)



Energy Gain/Loss

- The change in energy for a particle that moves from A to B is given by:



$$\Delta E = q \int_0^L \bar{E}_F(\bar{r}, t) \cdot d\bar{s} = qV$$

- Now define a voltage V such that V depends only on the particle trajectory. It includes the contribution of every electric field (RF fields, space charge fields, fields due to the interaction with the vacuum chamber, ...)
- In addition there are changes in energy $U(E)$ that depend also on the particle energy (e.g. radiation emitted by a particle under acceleration - synchrotron radiation)
- The total change in energy is given by the sum of the two terms:

$$\Delta E_T = qV + U(E)$$

Rate of Energy Change

The energy variation for the reference particle is given by:

$$\Delta E_T(s_0) = qV(s_0) + U(E_0)$$

For particle with energy $E = E_0 + \Delta E$ and orbit position $s = s_0 + \Delta s$:

$$\Delta E_T(s) = qV(s_0 + \Delta s) + U(E_0 + \Delta E) \cong qV(s_0) + q \left. \frac{dV}{ds} \right|_{s_0} \Delta s + U(E_0) + \left. \frac{dU}{dE} \right|_{E_0} \Delta E$$

Where the last expression holds for the case where
 $\Delta s \ll L_0$ (reference orbit length) and $\Delta E \ll E_0$.

In this approximation we can express the average rate of change of the energy respect to the reference particle energy by:

$$\frac{d\Delta E}{dt} \cong \frac{\Delta E_T(s) - \Delta E_T(s_0)}{T_0}$$



$$\frac{d\Delta E}{dt} \cong \frac{1}{T_0} \left(q \left. \frac{dV}{ds} \right|_{s_0} \Delta s + \left. \frac{dU}{dE} \right|_{E_0} \Delta E \right)$$

where $T_0 = \frac{L_0}{\beta_0 c}$ with $L_0 =$ length of the reference orbit between A and B
 $\beta_0 c =$ velocity of the reference particle

Longitudinal Phasespace

We just found:

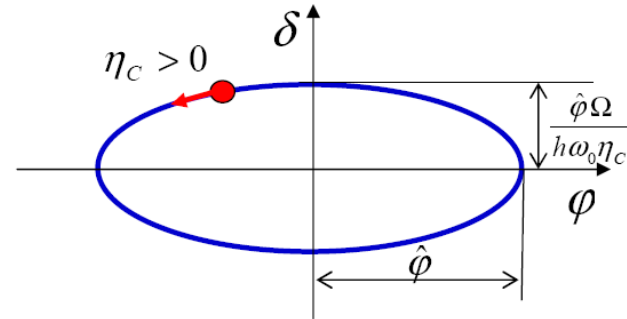
$$\varphi = \hat{\varphi} \cos(\Omega t + \psi)$$

$$\delta = \frac{\hat{\varphi} \Omega}{h \omega_0 \eta_c} \sin(\Omega t + \psi)$$



$$\frac{\varphi^2}{\hat{\varphi}^2} + \delta^2 \left(\frac{h \omega_0 \eta_c}{\hat{\varphi} \Omega} \right)^2 = 1$$

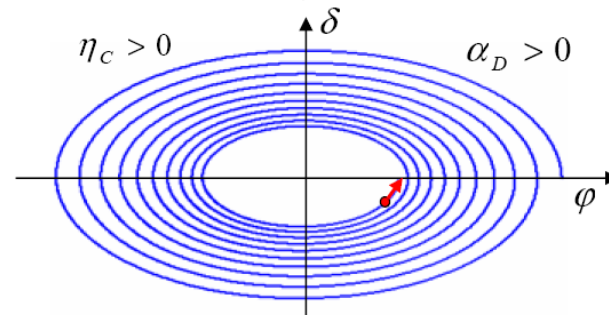
This equation represents an ellipse in the longitudinal phase space $\{\varphi, \delta\}$



With damping:

$$\varphi = \hat{\varphi} e^{-\alpha_D t} \cos(\Omega t + \psi)$$

$$\delta = \frac{\hat{\varphi} \Omega}{h \omega_0 \eta_c} e^{-\alpha_D t} \sin(\Omega t + \psi)$$



In rings with negligible synchrotron radiation (or with negligible non-Hamiltonian forces, the longitudinal emittance is conserved.

This is the case for heavy ion and for most proton machines.

Large Amplitudes/Separatrix

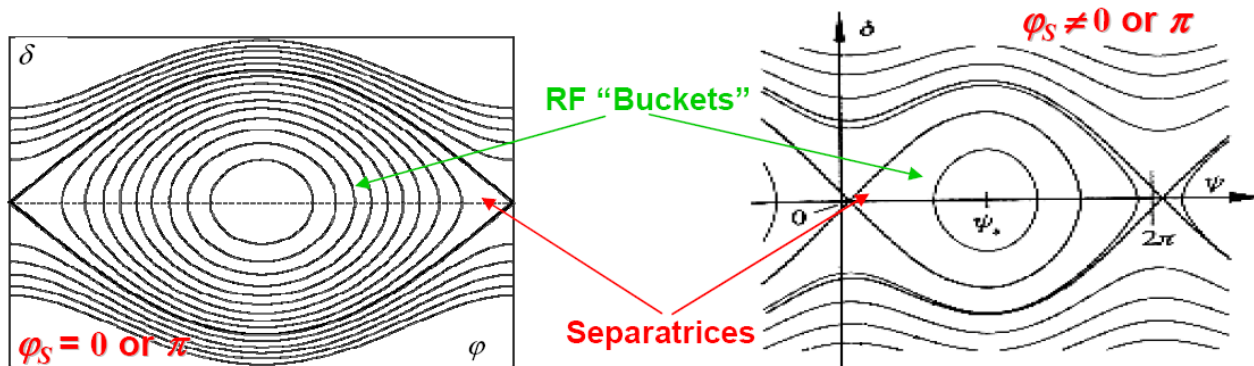
So far we have used the *small oscillation approximation* where:

$$\Delta E_T(\psi) = qV(\varphi_S + \varphi) = q\hat{V} \sin(\varphi_S + \varphi) \cong qV(\varphi_S) + q \left. \frac{dV}{d\varphi} \right|_{\varphi_S} \varphi = q\hat{V}\varphi_S + q\hat{V}\varphi$$

In the more general case of larger phase oscillations:

$$\Delta E_T(\psi) = qV(\varphi_S + \varphi) \cong q\hat{V} \sin(\varphi_S + \varphi)$$

And by Numerical integration:



- For larger amplitudes, trajectories in the phase space are not ellipsis anymore.
- Stable and unstable orbits exist. The two regions are separated by a special trajectory called *separatrix*
 - Larger amplitude orbits have smaller synchrotron frequencies

Radiation damping

Energy damping:

$$\alpha_D > 0 \quad \alpha_D = -\frac{1}{2T_0} \left. \frac{dU}{dE} \right|_{E_0}$$



$$\left. \frac{dU}{dE} \right|_{E_0} < 0$$

Larger energy particles lose more energy

$$P_{SR} = \frac{2}{3} \alpha h c^2 \frac{\gamma^4}{\rho^2}$$

- Typically, synchrotron radiation damping is very efficient in electron storage rings and negligible in proton machines.
- The damping time $1/\alpha_D$ (\sim ms for e-, \sim 13 hours LHC at 7 TeV) is usually much larger than the period of the longitudinal oscillations $1/2\pi\Omega$ (\sim μ s). This implies that the damping term can be neglected when calculating the particle motion for $t \ll 1/\alpha_D$:

$$\frac{d^2 \Delta s}{dt^2} + \Omega^2 \Delta s = 0$$

Harmonic oscillator equation